Asymptotic analysis for the solution to the Helmholtz problem in the exterior of a finite thin straight wire

Xavier Claeys

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Asymptotic analysis for the solution to the Helmholtz problem in the exterior of a finite thin straight wire

Xavier Claeys

N° 6277

Juillet 2007

Thème NUM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--6277--FR+ENG

Asymptotic analysis for the solution to the Helmholtz problem in the exterior of a finite thin straight wire

Xavier Claeys

Thème NUM — Systèmes numériques
Projets POems

Rapport de recherche n° 6277 — Juillet 2007 — 66 pages

Abstract: In this document we are interested in the solution of the Helmholtz equation with Dirichlet boundary condition in the exterior of a thin elongated body. We suppose that the geometry is well described in ellipsoidal coordinates. We propose an asymptotic analysis of this problem, using matched expansions. This leads to the construction of an approximate field with more explicit expression. The approximate field is composed of the first terms of the asymptotic expansion of the exact solution. Our study also leads to a validation of an acoustic version of the Pocklington's equation.

Key-words: Helmholtz, asymptotic, Pocklington, wire, thin, matched expansions, elongated

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

Asymptotic analysis for the solution to the Helmholtz problem in the exterior of a finite thin straight wire

Résumé : Dans ce document nous nous intéressons à la solution de l'équation de Helmholtz avec condition de Dirichlet à l'extérieur d'un corps fin élancé figurant un fil. Nous supposons que la géométrie peut-être convenablement décrite en coordonnées éllipsoidales. Nous proposons une analyse asymptotique de ce problème, en utilisant des développements raccordés. Ceci mène à la construction d'un champ approché dont l'expression est plus explicite et qui est composé des premiers termes du développement du champ exact. Notre étude mène également à la validation d'une version acoustique de l'équation de Pocklington.

Mots-clés : Helmholtz, asymptotique, Pocklington, fil, fin, développements raccordés, slender

1 Introduction

This document provides a study on the diffraction problem of an acoustic wave scattered by a thin wire. In order to be more precise, let us consider a domain containing, on the one hand, fixed obstacles denoted Ω_{ob}, and on the other hand, a thin elongated body Ω_ε^i with the shape of a wire (ε being the thickness of this wire). We wish to study the behavior of the solution to the Helmholtz problem with homogeneous Dirichlet boundary conditions in $\Omega_\varepsilon = \mathbb{R}^3 \setminus \overline{\Omega_{ob} \cup \Omega_\varepsilon^i}$, the source being given by a function f which support supp$f$ is disjoint from Ω_ε^i. This geometry is represented in the picture below.

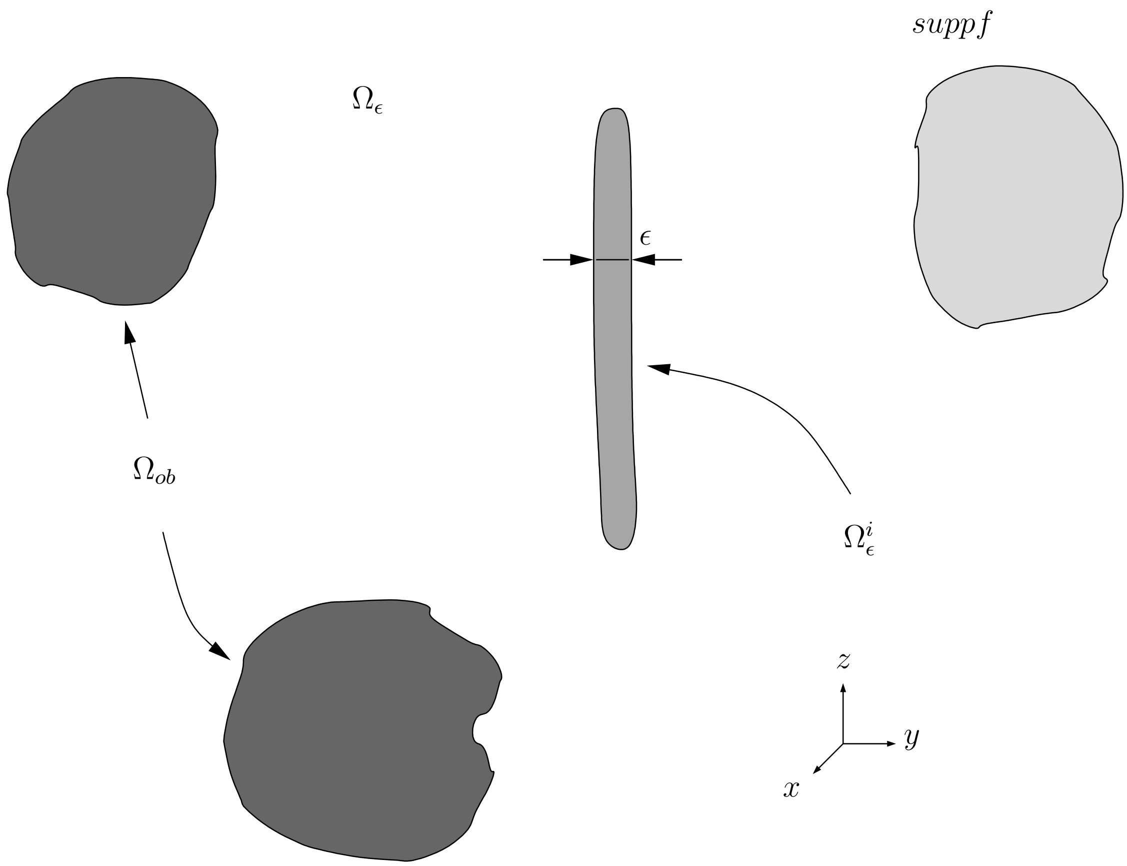


Figure 1: The geometry we consider

The problem of diffraction by a wire is an old problem that dates back to the end of the nineteenth century. In the physics literature , we have to cite the article of Pocklington [13]. It is a founding article for wire models. It derives a one dimensional integral equation taking into account the coupling between the current on a wire and the electromagnetic field in the exterior of this wire. We have also to cite the article by R.Holland by L.Simpson [21]. It introduces a very simple numerical scheme able to to take into account thin wires in a finite difference scheme for the computation of the propagation of an electromagnetic field. This method has been widely used so far, however it requires to determine empirically the value of a parameter called lineic inductance, and this raises many problems that led us to the present study. Finally we can cite the book of Taflov [2] or D.S.Jones [3]. They are interested in the numerical resolution of Maxwell's equations in geometries containing wires using the Pocklington equation. It has to be noticed that the model of Pocklington and Holland have not yet received a full theoretical justification. To the best of our knowledge, the Holland model has not been proved to be consistant so far.

Concerning mathematical studies on this subject, a first work has been realized by Fedoryuk in [8] and [18]. Fedoryuk proposes an asymptotic model on the basis of an integral equation in order to solve the Laplace problem, in a geometry with symmetry of revolution. This work has been followed by two complementary articles [12] and [10], that consider geometries with symmetry of revolution and coercive problems with both Dirichlet and Neuman boundary condition. A similar work has been simultaneously provided by F.Rogier [9]. This time the geometry has not necessarily the symmetry of revolution, however error estimates are less precise. The thesis of A.Mazari [1] takes into account the work of Rogier and completes it with numerical considerations on the Pocklington's equation for Maxwell's equations in a simple geometrical setting (straight cylindrical antenna in free space).

As was suggested by these articles, we want to tackle this problem using asymptotic analysis. For a didactic introduction to asymptotic analysis, the book by J.Sanchez-Hubert and E.Sanchez-Palencia [4] seems a good starting point. Among the possible approach we decided to use matched asymptotics. But there exists also other possible approaches such as the multiscale expansion technique. Matched asymptotics and multiscale expansions are equivalent as was shown in [7]. The choice of matched asymptotics is a first difference between the present work, and the work contained in the papers cited above. A study of general problems, restricted though to cases where maximum principle can be applied, is available in [24]. The book by Il'in [14] contains a great number of studies that are the occasion of presenting the application of matched asymptotics in a wide variety of situations. Also the book of Mazy'a, Nazarov and Plamenevskii contains the asymptotic study of many situations in which singularities appear via geometry (which is not always the case in problems concerned by asymptotics). This book proposes a study comparable to the one presented in Fedoryuk's article [18]. Mazy'a, Nazarov and Plamenevskii use multiscale expansions. Finally we would like to cite the two PhD thesis of S.Tordeux [22] an G.Vial [23] that are very instructive and represent good introductions both to mathched asymptotics an multiscale expansions.

We think that the present work contains innovative aspects in the way that we adopted to tackle the problem of thin wires. Concerning technical aspects, we have used matched asymptotics instead of multiscale expansions. In particular, we have adapted the work of Fedoryuk to a non-coercive situation (Helmholtz equation) which raised some technical difficulties. The ideas of section 5.3 are very similar to [18]. In error estimates, the proof of stability is classical in the literature of asymptotics. Concerning consistency, we have adapted technical results contained in [22] in the case of spheroidal coordinates (corollary 7.1 and lemma 7.5). Moreover section 8 come to an agreement with ideas contained in [1] (restreining the space of Lagrange multipliers to elements with symmetry of revolution), but the way of deriving these ideas is different. In particular, we did not encouter in literature any proof of stability on a mixed formulation like the result of 8.2. We believe that the interesting point of this work is, on the one hand, the result 7.1 that proves the validity of the matched expansion, and on the other hand, the result 8.2 that leads to a simpler formulation of the problem than the initial one: conceeding an error in $O(\varepsilon)$ it is possible to consider that the fields are constant on each cross section of the wire. This provides a justification of the (scalar version of the) Pocklington equation. Moreover this work seems to provide a unifying theoretical setting for wire models.

1.1 Brief survey on wire models

Before strating with our main discution, we shall first give an overview of the existing models for thin wires. These models deal with Maxwell's equations, but they can be adapted for the treatment of other equations such as Laplace and Helmholtz equation. We will present these existing models for Helmholtz equation, but keep in mind that similar models can be proposed for Laplace and Maxwell's equation. Each of these models differs only by its equation corresponding to the boundary condition on the wire. Until the end of this sub-section we consider a straight thin wire with a surface Γ^{ε} described described in cylindrical coordinates by $r = \Phi^{\varepsilon}(\theta, z)$ for $z \in [-1, 1]$.

Pocklington's model

The founder of wire models is H.C Pocklington who proposed in 1897 an equation coupling the field diffracted by an antenna with the current at the surface of this antenna. To simplify, suppose that $\Omega_{ob} = \emptyset$ (antenna in free space). Denote J the median line in this wire, with a geometrical structure as in figure 1, and suppose that this line is on the z-axis. Let $\gamma^\varepsilon(\theta, z)$ be the surface measure on Γ^ε in cylindrical coordinates. The length of the boundary of a section corresponding to the coordinate z is then

$$\widetilde{\gamma}^\varepsilon(z) = \int_0^{2\pi} \gamma^\varepsilon(\theta, z) d\theta$$

We consider an incident wave $u_{\mathrm{inc}} \in H^1_{loc}(\Omega^\varepsilon)$ satisfying the homogeneous Helmholtz equation in the neighborhood of the wire. For exemple we can contruct such a field by solving the equation $\Delta u_{\mathrm{inc}} + k^2 u_{\mathrm{inc}} = f$ with $\mathrm{supp} f \cap \Omega^i_\varepsilon = \emptyset$. We are interested in the corresponding diffracted wave , that is to say the function $u^\varepsilon_{\mathrm{d}} \in H^1_{loc}(\Omega_\varepsilon)$ satisfying:

$$\begin{cases} \Delta u^\varepsilon_{\mathrm{d}} + k^2 u^\varepsilon_{\mathrm{d}} = 0 \quad \text{in} \quad \Omega_\varepsilon \\ u^\varepsilon_{\mathrm{d}} = -u_{\mathrm{inc}} \quad \text{on} \quad \Gamma^\varepsilon \\ u^\varepsilon_{\mathrm{d}} \quad \text{outgoing} \end{cases}$$

Moreover we denote $u^\varepsilon_{\mathrm{tot}} = u^\varepsilon_{\mathrm{d}} + u_{\mathrm{inc}}$ the total field, and $G(\mathbf{x}; \mathbf{x}')$ the Green function of our acoustic problem and $I^\varepsilon = \widetilde{\gamma}^\varepsilon \frac{\partial u^\varepsilon_{\mathrm{d}}}{\partial n}_{|\Gamma^\varepsilon}$ that will represent the current on the wire. In order to write a relevant integral equation, Pocklington assumed that the current actually did not vary on a section of the wire, i.e $I^\varepsilon = I^\varepsilon(z)$ does not depend on z, and that the diffracted field in this region was also independant of z, $u^\varepsilon_{d\,|\Gamma^\varepsilon} = u^\varepsilon_{d\,|\Gamma^\varepsilon}(z)$. In the case of the Helmholtz problem, the integral equation of Pocklington is

$$\int_{z'=-1}^{+1} \Big(\int_{\theta=0}^{2\pi} \int_{\theta'=0}^{2\pi} \frac{e^{ik|\mathbf{x}(\theta,z)-\mathbf{x}(\theta',z')|}}{4\pi|\mathbf{x}(\theta,z)-\mathbf{x}(\theta',z')|} \, \frac{\gamma^\varepsilon(\theta,z)}{\widetilde{\gamma}^\varepsilon(z)} \, \frac{\gamma^\varepsilon(\theta,z')}{\widetilde{\gamma}^\varepsilon(z')} \, d\theta d\theta' \Big) \, I^\varepsilon(z') dz' = -u_{\mathrm{inc}\,|\Gamma^\varepsilon}(z).$$

We denoted $\mathbf{x}(\theta, z)$ the point of Γ^ε with cylindrical coordinates θ and z. This equation is a one dimensional version of the integral equation corresponding to a single layer potential used for the resolution of the Helmholtz problem. This equation is what we call the model of Pocklington.

Model of Pocklington with regularized kernel

The kernel of Pocklington's integral equation is singular (as a function of z'). Its singularity is logarithmic. Serveral authors have replaced it by a regularized kernel. Let us denote by $\widetilde{\mathbf{x}}(z)$ the point of J corresponding to the coordinate z. These authors have prefered to consider the equation

$$\int_{z'=-1}^{+1} \Big(\int_{\theta'=0}^{2\pi} \frac{e^{ik|\widetilde{\mathbf{x}}(z)-\mathbf{x}(\theta',z')|}}{4\pi|\widetilde{\mathbf{x}}(z)-\mathbf{x}(\theta',z')|} \gamma^\varepsilon(\theta',z') d\theta' \Big) \, I^\varepsilon(z') dz' = -u_{\mathrm{inc}\,|\Gamma^\varepsilon}(z)$$

Since the kernel of this equation is very smooth, the associated integral operator is compact and this new integral equation is ill-posed. As a consequence this equation can lead to realistic results only if the smallest mesh step used for the discretization is more than three times larger than the radius ε of the wire. This equation is what we call the model of Pocklington with regularized kernel. For the design of the code Magellan which purpose is to study conducting wires and shells, Bendali & al. have proposed a variational formulation in which the homogeneous Dirichlet boundary condition on the wire is taken into account using a one dimensional space of Lagrange multipliers (containing elements that do not depend on the θ variable). This restricted space is

$$H^{-1/2}_{\mathrm{moy}}(\Gamma^\varepsilon) = \{ q \in H^{-1/2}(\Gamma^\varepsilon) \ | \ \frac{\partial q}{\partial \theta} = 0 \ \}$$

where derivatives have to be understood in the sense of distributions. For the total field (incident+diffracted) an "averaged" Dirichlet boundary condition is then written on the median line

$$\int_{-1}^{+1} u^{\varepsilon}_{\mathrm{tot}}(\widetilde{\mathbf{x}}(z))\, q(z)\, dz = 0 \quad \forall\, q \in H^{-1/2}_{\mathrm{moy}}(\Gamma^{\varepsilon}).$$

Writing an integral representation for $u^{\varepsilon}_{\mathrm{tot}}$, this equation leads to a variational formulation of the regularized version of the Pocklington's equation. We call this model the regularized fictitious domain model. Mazari used this idea but tried to impose a Dirichlet boundary condition using an integral equation with a singular kernel, that correponds to the model of Pocklington. This equation writes

$$\int_{0}^{2\pi}\int_{-1}^{+1} u^{\varepsilon}_{\mathrm{tot}}(\mathbf{x}(\theta,z))\, q(z)\, dz\, d\theta = 0 \quad \forall\, q \in H^{-1/2}_{\mathrm{moy}}(\Gamma^{\varepsilon}).$$

We call this model, the singular fictious domain model.

Model of Holland

In order to solve numerically the problem of diffraction by an antenna, Holland & al. in [21] have proposed simple modifications of a classical FDTD scheme sufficient for simulating the presence of an antenna in the medium of propagation. As was proved by Collino and Millot in [5], the new resulting scheme can be inherited from a continuous variational formulation of the Maxwell's equations that differs from the usual formulation by the presence of an averaging operator and a parameter. This model assumes that the section of the wire is circular, the variations of the current on a transverse section are negligible, and the field has an electrostatic behavior in the neighborhood of the wire. From these assumptions, Holland deduces that $u^{\varepsilon}_{\mathrm{tot}}(r,\theta,z) \simeq I^{\varepsilon}(\theta,z)\ln(\frac{r}{\varepsilon})$. Then a function δ^{ε} has to be chosen in order to regularize the surface measure on Γ^{ε}. This function has to verify

$$\begin{array}{l} \int_0^{+\infty} \delta^{\varepsilon}(r) 2\pi r\, dr = 1, \\ \forall \varphi \text{ continuous}, \ \int \varphi(x,y)\delta^{\varepsilon}(\sqrt{x^2+y^2})\, dxdy \underset{\varepsilon\to 0_+}{\to} \varphi(0), \\ \delta^{\varepsilon}(r) = 0, \quad r_{\varepsilon} > r > \varepsilon \quad \text{and} \quad r^{\varepsilon} \underset{\varepsilon\to 0_+}{\to} 0. \end{array}$$

The choice of the function δ^{ε} lies on practical considerations. Instead of the usual homogeneous Dirichlet condition on the wire, one writes an equation averaged by δ^{ε} and makes use of the radial electrostatic behavior of the field. So the variational formulation that has to be discretized is

$$\left\{ \begin{array}{l} \displaystyle\int_{\Omega_R} \nabla u^{\varepsilon}_{\mathrm{tot}}\cdot\nabla\overline{v} - k^2\int_{\Omega_R} u^{\varepsilon}_{\mathrm{tot}}\overline{v} + \int_{\Gamma_R}\overline{v}T_R u^{\varepsilon}_{\mathrm{tot}} + \int_{\Omega_R}\delta^{\varepsilon}(r)I^{\varepsilon}(z)\overline{v}(r,\theta,z) = -\int_{\Omega_R} f\overline{v} \quad \forall v \in H^1(\Omega_R), \\ \displaystyle\int_{\Omega_R}\delta^{\varepsilon}(r)q(z)u^{\varepsilon}_{\mathrm{tot}}(r,\theta,z) = \int_{\Omega_R}\delta^{\varepsilon}(r)q(z)I^{\varepsilon}(z)\ln(\frac{r}{\varepsilon}) \quad \forall q \in H^{-1/2}_{\mathrm{moy}}(\Gamma^{\varepsilon}). \end{array}\right.$$

where T_R is a usual DTN map acting on the boundary of Ω_R.

1.2 Method of matched asymptotic expansions

We are now going to briefly present the matched asymptotic expansions method and the way we wish to apply it to our problem. This will give an outline of our study. In this document we are interested in the expansion with respect to ε of the function u^{ε} satisfying

$$\left\{ \begin{array}{l} \Delta u^{\varepsilon} + k^2 u^{\varepsilon} = f \quad \text{in} \quad \Omega_{\varepsilon}, \\ u^{\varepsilon} = 0 \quad \text{on} \quad \partial\Omega_{\varepsilon}, \\ u^{\varepsilon} \quad \text{outgoing}. \end{array}\right.$$

In order to study the behavior of u^ε we can first ask ourselves whether u^ε admits a limit (as $\varepsilon \to 0$). As the wire shrinks and disappears, one might guess that this limit exists and is the function u_0 satisfying

$$\begin{cases} \Delta u_0 + k^2 u_0 = f \quad \text{in} \quad \Omega, \\ \\ u_0 \quad \text{outgoing.} \end{cases}$$

Then one can ask what is the behavior of $u^\varepsilon - u_0$,... etc. We will then seek the first term of the expansion of u^ε. For this purpose, we will proceed in 5 steps

- **step 0**
  We describe in detail the geometrical setting of our problem, rigorously formulate the diffraction problem using variational formulations, and introduce a suitable functional setting. We will also define two regions $\mathcal{Z}_\mathrm{n}^\varepsilon$ and $\mathcal{Z}_\mathrm{f}^\varepsilon$ called near and far field regions. We will postulate a form in each of these regions for the first term of the expansion of u^ε:
  $$u^\varepsilon \simeq u_0 + u_1^\varepsilon \quad \text{in} \quad \mathcal{Z}_\mathrm{f}^\varepsilon \quad \text{with} \quad u_1^\varepsilon = o(u_0) \tag{1}$$
  We will call u_0 and u_1^ε "far field terms". For the near field region, as we don't really know the behavior of u^ε, We will simply write
  $$u^\varepsilon \simeq U_1^\varepsilon \quad \text{in} \quad \mathcal{Z}_\mathrm{n}^\varepsilon \tag{2}$$
  We will call U_1^ε "near field term".

- **step 1**
  This step is treated in section 3, where we seek in a formal and intuitive manner necessary conditions for $u_0 + u_1^\varepsilon$ to be a good approximation of u^ε in $\mathcal{Z}_\mathrm{lo}^\varepsilon$. We will caracterize $u_0 + u_1^\varepsilon$ up to the choice of a function $a^\varepsilon \in L^2([-1;1])$. The definition of a^ε is delayed to section 5.

- **step 2**
  This step is treated in section 4 where we seek necessary conditions for U_1^ε to be an approximation of u^ε in $\mathcal{Z}_\mathrm{pr}^\varepsilon$. We will caracterize U_1^ε up to the choice of a function $b^\varepsilon \in L^2([-1;1])$.

- **step 3**
  To end the definition of a suitable approximate field we need to define a^ε and b^ε. This will be the subject of this step treated in section 5 where we prove that these functions have to satisfy an integral equation.

- **step 4**
  We would have *formally* constructed $u_0 + u_1^\varepsilon$ and U_1^ε in the preceding step, so this step will deal with defining an approximate field $\widetilde{u}^\varepsilon$ using the far and near field (section 6), and then prove in section 7 that $\|\widetilde{u}^\varepsilon - u^\varepsilon\|_{H^1(\Omega_R)}$ is small, which is the subject of theorem theorem 7.1 that states
  $$\|u^\varepsilon - \widetilde{u}^\varepsilon\|_{H^1(\Omega_R)} \leqslant \kappa\, \varepsilon^{1/2} \ln 1/\varepsilon$$
  for a constant $\kappa > 0$ independant of ε.

Once we have applied this method, we would have obtained an approximation of the exact solution. The question is what to do with this approximation. We will use it to prove that u^ε is also very close to another function u^ε that is solution to the same diffraction problem with simplified boundary conditions. This problem will be written using a mixed formulation that we call the simplified fictitious domain formulation. We will prove a result very similar to 7.1 with u^ε, which will yield a validation of the model of Pocklington.

2 Step 0: geometrical setting and initial problem

2.1 General remarks

Consider a space of reference $\mathcal{E} = \mathbb{R}^3$. So $\mathcal{E}$ has a natural manifold structure. When we write "$\mathbf{x}$" we refer to a point of $\mathcal{E}$ as a manifold, thus independantly of any coordinate system. We will often refer to objects related to $\mathcal{E}$ and so independant of any coordinate system, such as functions for which we write $u(\mathbf{x})$.

We wish to emphasize the difference between the manifold structure of $\mathcal{E}$, and its representation in coordinate systems. We will refer to a point $\mathbf{x} \in \mathcal{E}$ with 3 coordinates chosen in a set different from $\mathcal{E}$. For exemple we will naturally refer to a point using its cartesian coordinates $(x; y; z) \in \mathbb{R}^3$. We will use four coordinate systems. The first is the cartesian system with coordinates denoted $(x; y; z)$. We will write $\mathbf{x}(x; y; z)$ in order to mean that the cartesian coordinates $\mathbf{x}$ are $(x; y; z)$. The second system is the spherical coordinates restricted to the unit sphere, with coordinates denoted (ν, θ) (be careful we don't use the usual coordinates (θ, φ) in the present case). The third system is called in litterature prolate spheroidal coordinates. It belongs to the family of ellipsoidal coordinates, so we call it ellipsoidal in order not to make any confusion with spherical coordinates. The coordinates associated with this system are denoted $(\xi; \nu; \theta)$. In accordance with our notation for cartesian coordinates, we write $\mathbf{x}(\xi; \nu; \theta)$ to mean that $\mathbf{x}$ has $(\xi; \nu; \theta)$ as ellipsoidal coordinates. We finally consider a last coordinate system that we call scaled coordinates system. It will be defined below. The corresponding coordinates are denoted (ζ, ν, θ). Again we write $\mathbf{x}(\zeta; \nu; \theta)$ in order to mean that $\mathbf{x}$ has $(\zeta; \nu; \theta)$ for scaled coordinates.

Sometimes we will write "$u(\xi; \nu; \theta)$" instead of writing "$u(\mathbf{x})$ where $\mathbf{x}$ has $(\xi; \nu; \theta)$ for ellipsoidal coordinates" (of course the same remark is true for cartesian, spherical or scaled coordinates). Also when considering a function depending only on one variable , let say u depending only on ξ, then we simply write "$u(\xi)$" instead of "$u(\mathbf{x})$ where $\mathbf{x}$ has $(\xi; \nu; \theta)$ for ellipsoidal coordinates."

Now we recall the precise definition of our coordinate systems, and define the scaled coordinates.

2.2 Coordinate systems

In this sub-section, we shall remind the reader some definitions concerning usual cordinate systems, and also introduce some others. As we stated before, we will be interested in spherical coordinates, ellipsoidal coordinates, and a new coordinate system called scaled coordinates.

2.2.1 Spherical coordinates

In this document we will consider coordinates on the unit sphere $S^2 = \{\mathbf{x} \in \mathcal{E} \text{ t.q } |\mathbf{x}| = 1\}$ slightly different from usual spherical coordinates. Let us denote these coordinates (ν, θ). One such pair will refer to a point on S^2 with cartesian coordinates given by the formula

$$\begin{cases} x = \sqrt{1-\nu^2}\cos\theta & \nu \in [-1; +1], \\ y = \sqrt{1-\nu^2}\sin\theta & \theta \in [0; 2\pi], \\ z = \nu. \end{cases}$$

We denote $O_+^{S^2}$ and $O_-^{S^2}$ the upper and lower poles of S^2. We also denote $\phi_{\mathrm{sp}}^{S^2} : S^2 \setminus \{O_+^{S^2}, O_-^{S^2}\} \to]-1, +1[\times (\mathbb{R}/2\pi\mathbb{Z})$ the chart that associates to any point $\mathbf{x} \in S^2$ its spherical coordinates (ν, θ) given by the above formula. $\phi_{\mathrm{sp}}^{S^2}$ is a C^∞-difféomorphism. Here is the expression of the Laplace operator on the unit sphere,

$$\Delta_{S^2} = \frac{\partial}{\partial\nu}(1-\nu^2)\frac{\partial}{\partial\nu} + \frac{1}{1-\nu^2}\frac{\partial^2}{\partial\theta^2}.$$

2.2.2 Ellipsoidal coordinates

Let us recall how these coordinates $(\xi;\nu;\theta)$ are defined using cartesian coordinates,

$$\begin{cases} x = \sqrt{(\xi^2-1)(1-\nu^2)}\cos\theta & \xi \in [1;+\infty[\\ y = \sqrt{(\xi^2-1)(1-\nu^2)}\sin\theta & \nu \in [-1;+1] \\ z = \xi\,\nu & \theta \in [0;2\pi] \end{cases}$$

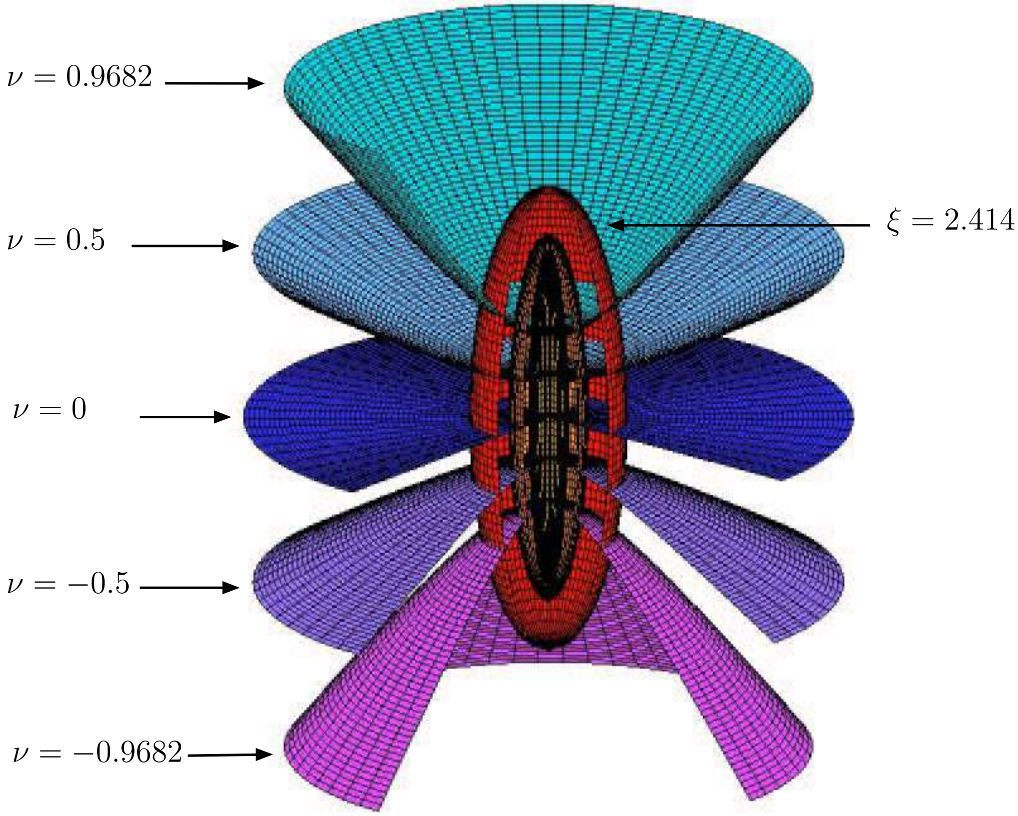
Let $U_{el} = \mathcal{E} \setminus \{\mathbf{x}(x,y,z) \mid x^2+y^2 = 0\ \}$. We denote $\phi_{el} : U_{el} \to]1,+\infty[\times]-1,+1[\times(\mathbb{R}/2\pi\mathbb{Z})$ the function that associates to any point its ellipsoidal coordinates. ϕ_{el} is a C^∞-difféomorphism. The iso-ξ and iso-ν surfaces corresponding to this coordinate system are represented on the figure below. Let us find the explicit equation of these surfaces. First we deal with a surface of equation $\xi = \xi_0$. It is possible to make ν and θ disappear in the formulas defining ellipsoidal coordinates. One then obtain

$$\frac{z^2}{\xi_0^2} + \frac{x^2+y^2}{\xi_0^2-1} = 1,$$

which is the equation of an ellipsoid of revolution. Now we treat the case of the surface of equation $\nu = \nu_0$. The same argument as before yields the equation

$$\frac{z^2}{\nu_0^2} - \frac{x^2+y^2}{1-\nu_0^2} = 1,$$

which is the equation of an hyperboloid of revolution.



Concerning the metric tensor we compute

$$(\frac{\partial x}{\partial \xi})^2 + (\frac{\partial y}{\partial \xi})^2 + (\frac{\partial z}{\partial \xi})^2 = \frac{\xi^2-\nu^2}{\xi^2-1},$$
$$(\frac{\partial x}{\partial \nu})^2 + (\frac{\partial y}{\partial \nu})^2 + (\frac{\partial z}{\partial \nu})^2 = \frac{\xi^2-\nu^2}{1-\nu^2},$$
$$(\frac{\partial x}{\partial \theta})^2 + (\frac{\partial y}{\partial \theta})^2 + (\frac{\partial z}{\partial \theta})^2 = (\xi^2-1)(1-\nu^2).$$

Moreover, as the coordinate lines are orthogonal, as is easily verified, we directly deduce the expression of the metric tensor and the measure in ellipsoidal coordinates

$$d^2m \;=\; \frac{\xi^2-\nu^2}{\xi^2-1}\,d^2\xi \;+\; \frac{\xi^2-\nu^2}{1-\nu^2}\,d^2\nu \;+\;(\xi^2-1)(1-\nu^2)\,d^2\theta \quad \text{and} \quad dxdydz \;=\; (\xi^2-\nu^2)\,d\xi d\nu d\theta$$

We also deduce the expression of laplacian and gradient operators,

$$\nabla\psi = \frac{\xi^2-1}{\xi^2-\nu^2}\frac{\partial\psi}{\partial\xi}\,\mathbf{e}_\xi + \frac{1-\nu^2}{\xi^2-\nu^2}\frac{\partial\psi}{\partial\nu}\,\mathbf{e}_\nu + \frac{1}{(\xi^2-1)(1-\nu^2)}\frac{\partial\psi}{\partial\theta}\,\mathbf{e}_\theta$$
$$\Delta\,\psi = \frac{1}{\xi^2-\nu^2}\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial\psi}{\partial\xi} + \frac{1}{\xi^2-\nu^2}\frac{\partial}{\partial\nu}(1-\nu^2)\frac{\partial\psi}{\partial\nu} + \frac{1}{(\xi^2-1)(1-\nu^2)}\frac{\partial^2\psi}{\partial\theta^2}.$$

where the vectors $\mathbf{e}_\xi$, $\mathbf{e}_\nu$ and $\mathbf{e}_\theta$ are defined using the cartesian vectors $\mathbf{e}_x, \mathbf{e}_y$ and $\mathbf{e}_z$ according to the relations

$$\mathbf{e}_\xi = \frac{\partial x}{\partial\xi}\,\mathbf{e}_x + \frac{\partial y}{\partial\xi}\,\mathbf{e}_y + \frac{\partial z}{\partial\xi}\,\mathbf{e}_z\;, \quad \mathbf{e}_\nu = \frac{\partial x}{\partial\nu}\,\mathbf{e}_x + \frac{\partial y}{\partial\nu}\,\mathbf{e}_y + \frac{\partial z}{\partial\nu}\,\mathbf{e}_z \text{ and } \quad \mathbf{e}_\xi = \frac{\partial x}{\partial\theta}\,\mathbf{e}_x + \frac{\partial y}{\partial\theta}\,\mathbf{e}_y + \frac{\partial z}{\partial\theta}\,\mathbf{e}_z.$$

2.2.3 Scaled coordinates

The scaled coordinates of a point will be denoted $(\zeta;\nu;\theta)$ (compared to the cartesian coordinates only the first coordinate change) and will be defined using ellipsoidal coordinates by the formula

$$\xi^2 = 1 + \varepsilon^2\zeta^2$$

or by means of cartesian coordinates using the formulas

$$\begin{cases} x = \varepsilon\,\zeta\sqrt{1-\nu^2}\cos\theta & \zeta\in[0;+\infty[\\ y = \varepsilon\,\zeta\sqrt{1-\nu^2}\sin\theta & \nu\in]-1;+1[\\ z = \sqrt{1+\varepsilon^2\zeta^2}\,\nu & \theta\in[0;2\pi]. \end{cases}$$

The expression of the measure with these coordinates is $\frac{\varepsilon^2\zeta}{\xi}(\xi^2-\nu^2)d\zeta d\nu d\theta$.

2.3 Geometrical setting

2.3.1 Two remarks on the choice of the coordinate system

We have chosen the use of ellipsoidal coordinates in order to describe the geometrical elements of our problem because neither the spherical system nor the cylindrical one are well suited to the geometry of the tips. Our choice for the scaled coordinates comes from the fact that the laplacian has a very simple expression with these coordinates, as we shall see in section 4.

2.3.2 Description of the geometry

We will now describe the geometry we want to consider for our analysis. First of all we wish to consider an open set $\Omega \subset \mathcal{E}$ that we assume to be C^∞. We will refer to the unit segment located between the points $(0;0;1)$ and $(0;0;-1)$ by the letter J, and assume that $J \subset \Omega$. In ellipsoidal coordinates, the segment J is described by the equation $\xi = 1$. We will also consider an open set Ω_ε^i with a boundary Γ^ε described in ellipsoidal coordinates by the equation

$$(\Gamma^\varepsilon): \quad \xi^2 = 1 + \varepsilon^2\Phi^2(\nu;\theta) \quad , \quad \nu \in [-1;+1] \text{ and } \theta \in [0;2\pi].$$

This equation written in scaled coordinates becomes very simple

$$(\Gamma^\varepsilon): \quad \zeta = \Phi(\nu;\theta) \quad , \quad \nu \in [-1;+1] \text{ and } \theta \in [0;2\pi].$$

This surface will represent the wire during our study. We make the following assumptions on Φ:

- **H 1 :** Φ is C^∞,
- **H 2 :** There exists a constant Φ_0 such that $\Phi_0 < \Phi(\nu;\theta) \quad \forall\nu \in [-1;+1]\ ,\ \forall\theta \in [0;2\pi]$,
- **H 3 :** There exists a $\nu_0 \in]0;1[$ such that $\Phi(\nu;\theta) = \Phi(\nu)$ for $|\nu| > \nu_0$.

Note that hypothesis **H1** excludes the presence of geometrical singularities such as corners or edges at the surface of Γ^ε. Moreover hypothesis **H3** assumes that the wire has symmetry of revolution in the neighborhood of the tips. In spite of these restrictions, a certain variety of shapes for the wire can be represented by this type of equation if we suppose that ε is fixed.

Let us denote Ω_ε^i the interior set of Γ^ε, and denote $\Omega_\varepsilon = \Omega \setminus \overline{\Omega_\varepsilon^i}$. We also denote B_R the open ball of center 0 and radius R, and $\Gamma_R = \partial B_R$. We define $\Omega_\varepsilon^R = \Omega_\varepsilon \cap B_R$ and $\Omega_R = \Omega \cap B_R$. When $\varepsilon \to 0$, $\Omega_\varepsilon^i \to J$ in the sense of Haussdorf: the obstacle shrinks around the segment J.

We denote $O_+^{\Gamma^\varepsilon}$ and $O_-^{\Gamma^\varepsilon}$ the upper and lower poles on Γ^ε. Since the equation of Γ^ε is given in ellipsoidal coordinates by $(\Gamma^\varepsilon): \ \xi^2 = 1 + \varepsilon^2\Phi^2(\nu;\theta)$, we see that (ν,θ) is a good coordinate system on $\Gamma^\varepsilon \setminus \{O_+^{\Gamma^\varepsilon}, O_-^{\Gamma^\varepsilon}\}$. Let us denote $\phi_{\text{el}}^{\Gamma^\varepsilon} : \Gamma^\varepsilon \setminus \{O_+^{\Gamma^\varepsilon}, O_-^{\Gamma^\varepsilon}\} \to]-1,+1[\times(\mathbb{R}/2\pi\mathbb{Z})$ the chart that maps any point $\mathbf{x} \in \Gamma^\varepsilon$ with ellipsoidal coordinates $(\sqrt{1+\varepsilon^2\Phi^2(\nu,\theta)},\nu,\theta)$ to its coordinates (ν,θ). $\phi_{\text{el}}^{\Gamma^\varepsilon}$ is a C^∞-difféomorphism.

2.3.3 Several useful sets

During our analysis, we will need to refer to three particular sets, three zones in Ω_ε .

$$\mathcal{Z}_{\text{pr}}^\varepsilon = \{\mathbf{x}(\xi;\nu;\theta) \in \mathbb{R}^3 \mid \varepsilon^2\Phi^2(\nu;\theta) < \xi^2 - 1 < 2\varepsilon\},$$

$$\mathcal{Z}_{\text{lo}}^\varepsilon = \{\mathbf{x}(\xi;\nu;\theta) \in \mathbb{R}^3 \mid \varepsilon < \xi^2 - 1\},$$

$$\mathcal{T}^\varepsilon = \mathcal{Z}_{\text{pr}}^\varepsilon \cap \mathcal{Z}_{\text{lo}}^\varepsilon = \{\mathbf{x}(\xi;\nu;\theta) \in \mathbb{R}^3 \mid \varepsilon < \xi^2 - 1 < 2\varepsilon\}.$$

We will call $\mathcal{Z}_{\text{pr}}^\varepsilon$ the near field zone, and $\mathcal{Z}_{\text{lo}}^\varepsilon$the far field zone , and $\mathcal{T}^\varepsilon$ the transition zone. These zones are delimited by ellipsoids. $\mathcal{T}^\varepsilon$ is like an ellipsoidal shell shrinking around J. For purely practical reasons, we will also need to refer to an open neighborhood of J in Ω. We assume that there exists $\xi_0 > 1$ such that $\{\mathbf{x}(\xi;\nu;\theta) \in \mathbb{R}^3 \mid \xi < 3\xi_0\} \subset \Omega$. Then we define $\mathcal{O} = \{\mathbf{x}(\xi;\nu;\theta) \mid \xi < 2\xi_0\}$ and note that $J \subset \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$.

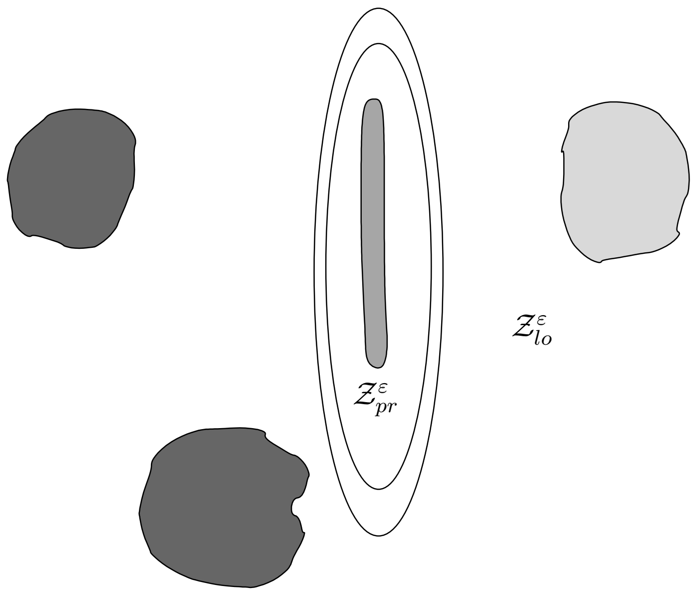


Figure 2: Far field and near field zones

2.4 Initial problem and functional setting

2.4.1 Standard space

We simply denote $\mathcal{H} = \{\, v \in H^1(\Omega_R)$ t.q $v_{|\partial\Omega} = 0 \,\}$. We define a sesquilinear form on $\mathcal{H}$ by $(u;v)_{\mathcal{H}} = \int_{\Omega_R} \nabla u \cdot \nabla \overline{v} + \int_{\Omega_R} u\,\overline{v}$. With this sesquilinear form, $\mathcal{H}$ is an Hilbert space. We will also refer to the operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ defined by $(\mathcal{A}u;v)_{\mathcal{H}} = \int_{\Omega_R} \nabla u \cdot \nabla \overline{v} - k^2 \int_{\Omega_R} u\,\overline{v} + \int_{\Gamma_R} \overline{v}\, T_R u \quad \forall u,v \in \mathcal{H}$. T_R is a usual Dirichlet-to-Neumann map for which we give the explicit expression and the properties in the appendix A. We also denot γ^{ε} the surface measure on Γ^{ε}. So if $v \in C^{\infty}(\Omega_R)$, then

$$\int_{\Gamma^{\varepsilon}} v(\mathbf{x})\, d\sigma(\mathbf{x}) = \int_{\nu=-1}^{+1} \int_{\theta=0}^{2\pi} v \circ (\phi_{\mathrm{el}}^{\Gamma^{\varepsilon}})^{-1}(\nu;\theta)\, \gamma^{\varepsilon}(\nu;\theta) d\nu\, d\theta.$$

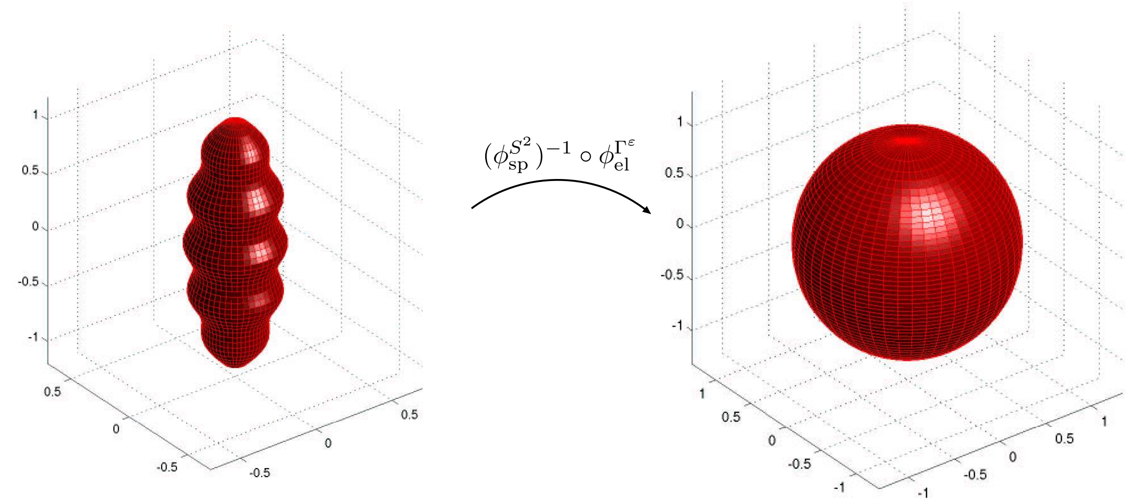
We denote $\mathcal{H}_0^{\varepsilon} = \{\, v \in \mathcal{H}$ t.q $v_{|\Gamma^{\varepsilon}} = 0 \,\}$. In §7.1 we use the operator $\mathcal{A}_0^{\varepsilon} : \mathcal{H}_0^{\varepsilon} \to \mathcal{H}_0^{\varepsilon}$ defined by $(\mathcal{A}_0^{\varepsilon} u;v)_{\mathcal{H}} = (\mathcal{A}u;v)_{\mathcal{H}} \quad \forall u,v \in \mathcal{H}_0^{\varepsilon}$.

2.4.2 Functional space on the wire

We will need a functional setting for functions "living on Γ^{ε}" that will in practice be function on J. First denote $\mathcal{L} = \frac{\partial}{\partial\nu}(1-\nu^2)\frac{\partial}{\partial\nu}$ unbounded operator on $L^2(I)$. Define $E^n(I) = \{u \in L^2(I) \mid \langle(-\mathcal{L})^k u; u\rangle_{L^2(I)} < +\infty \quad k = 0 \ldots n\}$ with the norm $\|u\|^2_{E^n(I)} = \sum_{q=0}^{n} \langle(-\mathcal{L})^q u; u\rangle_{L^2(I)}$.

2.4.3 Trace space

Now we will study in detail the space $H^{1/2}(\Gamma^{\varepsilon})$. We begin with a technical lemma of geometry that establishes topological equivalence between the wire Γ^{ε} and the sphere S^2. This enables us to study $H^{1/2}(S^2)$ instead of $H^{1/2}(\Gamma^{\varepsilon})$. In particular we will define an explicit norm on $H^{1/2}(\Gamma^{\varepsilon})$ using this equivalence.



Lemma 2.1
The map $(\phi_{\mathrm{sp}}^{S^2})^{-1} \circ \phi_{\mathrm{el}}^{\Gamma^\varepsilon} : \Gamma^\varepsilon \backslash \{O_+^{\Gamma^\varepsilon}, O_-^{\Gamma^\varepsilon}\} \to S^2 \backslash \{O_+^{S^2}, O_-^{S^2}\}$ *can be extended in a* C^∞*-diffeomorphism of* Γ^ε *into* S^2.

Proof:
Defining $(\phi_{\mathrm{sp}}^{S^2})^{-1} \circ \phi_{\mathrm{el}}^{\Gamma^\varepsilon}(O_+^{\Gamma^\varepsilon}) = O_+^{S^2}$ and $(\phi_{\mathrm{sp}}^{S^2})^{-1} \circ \phi_{\mathrm{el}}^{\Gamma^\varepsilon}(O_-^{\Gamma^\varepsilon}) = O_-^{S^2}$, it is easy to see that $(\phi_{\mathrm{sp}}^{S^2})^{-1} \circ \phi_{\mathrm{el}}^{\Gamma^\varepsilon}$ is a bijective map between Γ^ε and S^2. If we can prove the regularity of $(\phi_{\mathrm{sp}}^{S^2})^{-1} \circ \phi_{\mathrm{el}}^{\Gamma^\varepsilon}$ extended in this manner , then the inverse mapping theorem will yield the desired result. We construct a function $\mathcal{G} : \mathcal{E} \to \mathcal{E}$ such that $\mathcal{G}_{|\Gamma^\varepsilon} = (\phi_{\mathrm{sp}}^{S^2})^{-1} \circ \phi_{\mathrm{el}}^{\Gamma^\varepsilon}$ and show that $\mathcal{G}$ si a C^∞ function on a neighborhood of Γ^ε. Define $\phi = \mathrm{pr} \circ \phi_{el} \circ \phi_{\mathrm{car}}^{-1}$, where pr is the projection on the first ellipsoidal coordinate ξ, and ϕ_{car} is the map associated with cartesian coordinates. Thus for a point $\mathbf{x} \in \mathcal{E}$ with cartesian coordinates (x, y, z) and ellipsoidal coordinates (ξ, ν, θ), we have $\xi = \phi(x, y, z)$. We show that $\phi^2 - 1$ is a C^∞ function on $\phi_{\mathrm{car}}(\mathcal{E} \setminus J)$, and $\phi^2(x, y, z) - 1 > 0$ for any $(x, y, z) \in \phi_{\mathrm{car}}(\mathcal{E} \setminus J)$. Indeed, according to the formulas defining the ellipsoidal system, the following equality always holds

$$\frac{x^2 + y^2}{\xi^2 - 1} + \frac{z^2}{\xi^2} = 1. \tag{3}$$

So by means of a simple computation we obtain the explicit expression of ϕ,

$$\phi^2(x, y, z) - 1 = \frac{-(1 - x^2 - y^2 - z^2) + \sqrt{(1 - x^2 - y^2 - z^2)^2 + 4(x^2 + y^2)}}{2}.$$

This yields the desired properties. We also obtain the same properties for ϕ and $\sqrt{\phi^2 - 1}$. We define $\mathcal{G}$ in cartesian coordinates by

$$\phi_{\mathrm{car}} \circ \mathcal{G} \circ \phi_{\mathrm{car}}^{-1}(x, y, z) = \left(\frac{x}{\sqrt{\phi^2(x, y, z) - 1}}, \frac{y}{\sqrt{\phi^2(x, y, z) - 1}}, \frac{z}{\phi(x, y, z)}\right).$$

Using the equation (3), we see that $S^2 = \mathcal{G}(\mathcal{E} \setminus J)$ and that $\mathcal{G}$ is a C^∞ on $\mathcal{E} \setminus J$. We also easily see that $\mathcal{G}_{|\Gamma^\varepsilon} = (\phi_{\mathrm{sp}}^{S^2})^{-1} \circ \phi_{\mathrm{el}}^{\Gamma^\varepsilon}$, hence the result.

■

The preceding result leads to a very simple caracterization of $H^{1/2}(\Gamma^\varepsilon)$. Let us recall that Sobolev spaces on S^2 can be caracterized using spherical harmonics $P_l^m(\nu)e^{im\theta}$ where P_l^m are the associated Legendre functions. A detailed presentation of these functions is given in [19],chap.7. Note by the way that these functions are not normlized in $L^2(I)$ i.e. $\|P_l^m\|_{L^2(I)} \neq 1$. This is the reson why we introduce their normalized counterpart $\widetilde{P}_l^m = P_l^m/\|P_l^m\|_{L^2(I)}$.

Corollary 2.1

$$u \in H^{1/2}(\Gamma^\varepsilon) \quad \Leftrightarrow \quad \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}(1+l^2)^{1/2}\Big|\int_{\nu=-1}^{+1}\int_{\theta=0}^{2\pi} u(\nu,\theta)\widetilde{P}_l^m(\nu)e^{-im\theta}\,d\nu\,d\theta\Big|^2 < +\infty.$$

Note that the expression "$u(\nu,\theta)$" is not perfectly rigorous, and we should write "$u\circ(\phi_{\mathrm{el}}^{\Gamma^\varepsilon})^{-1}(\nu,\theta)$" instead. We can reformulate this corollary writing

$$u \in H^{1/2}(\Gamma^\varepsilon) \quad \Leftrightarrow \quad \big((\phi_{\mathrm{el}}^{\Gamma^\varepsilon})^{-1}\circ\phi_{\mathrm{sp}}^{S^2}\big)^* u \in H^{1/2}(S^2). \tag{4}$$

Here we have used the notation $*$ defined by $\phi^* u = u\circ\phi$. The preceding caracterization depends on ε only through the chart $\phi_{\mathrm{el}}^{\Gamma^\varepsilon}$. From this caracterization, we obtain a natural norm

$$\|u\|^2_{H^{1/2}(\Gamma^\varepsilon)} = \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}(1+l^2)^{1/2}\,\Big|\int_{\nu=-1}^{+1}\int_{\theta=0}^{2\pi} u(\nu,\theta)\widetilde{P}_l^m(\nu)e^{-im\theta}\,d\nu\,d\theta\Big|^2.$$

We also define the dual norm in the following manner

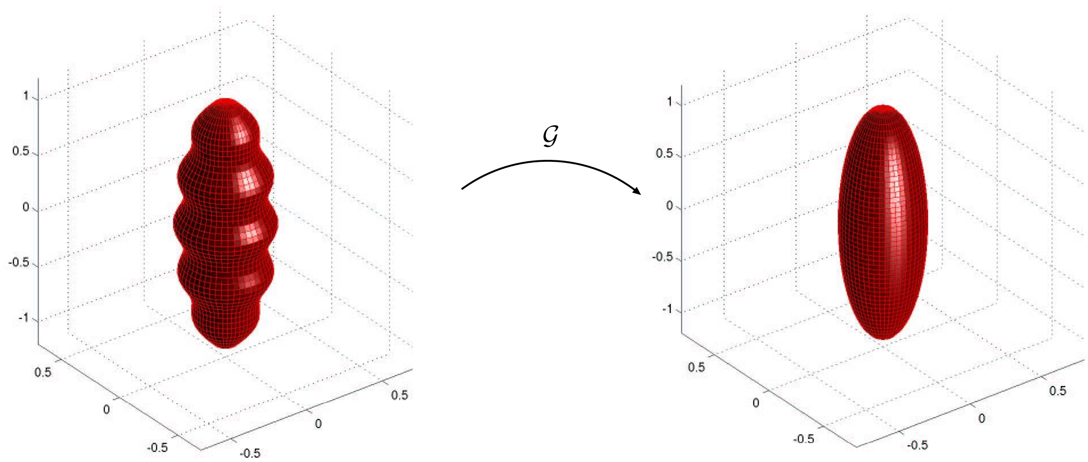
$$\|u\|_{H^{-1/2}(\Gamma^\varepsilon)} = \sup_{v\in H^{1/2}(\Gamma^\varepsilon)}\frac{\int_{\Gamma^\varepsilon} u\overline{v}}{\|v\|_{H^{1/2}(\Gamma^\varepsilon)}}.$$

We will now construct a lifting operator continuously mapping $H^{1/2}(\Gamma^\varepsilon)$ to $H^1(\Omega_R)$, with an operator norm bounded as $\varepsilon \to 0$.

Lemma 2.2
There exists $\kappa, \varepsilon_0 > 0$ such that $\forall\varepsilon \in]0,\varepsilon_0[$, $\forall u \in H^{1/2}(\Gamma^\varepsilon)$ there exists $\mathcal{R}(u) \in H^1(\Omega_R)$ satisfying $\mathcal{R}(u)_{|\Gamma^\varepsilon} = u$ and $\|\mathcal{R}(u)\|_{H^1(\Omega_R)} \leqslant \kappa\|u\|_{H^{1/2}(\Gamma^\varepsilon)}$.

Proof:
To prove this lemma we will transform our actual wire into an ellipsoid which has a simpler (separable) shape. Denote $\breve{\Gamma}^\varepsilon$ the ellipsoid with equation $\xi^2 = 1+\varepsilon^2$. We introduce a geometrical transformation mapping Γ^ε into the ellipsoid $\breve{\Gamma}^\varepsilon$. Let $\mathcal{G}:\mathcal{E}\to\mathcal{E}$ be a bijective function (take care that this $\mathcal{G}$ has nothing to do with the $\mathcal{G}$ of lemma 2.1). $\mathcal{G}$ is described in ellipsoidal coordinates by the formula

$$\mathcal{G}_{\mathrm{el}}(\xi',\nu',\theta') = \phi_{\mathrm{el}}\circ\mathcal{G}\circ\phi_{\mathrm{el}}^{-1}(\xi',\nu',\theta') = (\xi,\nu,\theta) = (\sqrt{1+\frac{\xi'^2-1}{\Phi^2(\nu',\theta')}}\,,\,\nu',\theta')$$



coordinates: (ξ', ν', θ') coordinates: (ξ, ν, θ)

Note first that $\mathcal{G}(\Gamma^\varepsilon) = \breve{\Gamma}^\varepsilon$, and also $(\mathcal{G}^{-1})_{|\breve{\Gamma}^\varepsilon} = (\phi_{\rm el}^{\Gamma^\varepsilon})^{-1} \circ \phi_{\rm el}^{\breve{\Gamma}^\varepsilon}$. Using the ξ_0 of §2.3.3, we introduce fixed numbers $1 < \xi'_* < \xi_0$ and $1 < \xi_* < \xi_0$ sufficiently close to 1 and such that the relation $\xi \leqslant \xi_* \Rightarrow \xi' \leqslant \xi'_*$ is satisfied. We start by showing the existence of a $\kappa > 0$ such that for any $u \in H^1(\Omega_R)$ satisfying $u(\xi, \nu, \theta) = 0$ when $\xi < \xi_*$, we have $u \circ \mathcal{G} \in H^1(\Omega_R)$ with $\|u \circ \mathcal{G}\|_{H^1(\Omega_R)} \leqslant \kappa \|u\|_{H^1(\Omega_R)}$. Indeed in ellipsoidal coordinates, the jacobian of the transformation $\mathcal{G}_{el}^{-1}$ is given by

$$Jac\,(\mathcal{G}_{el}^{-1}) = \frac{\xi\Phi^2}{\sqrt{1 + (\xi^2 - 1)\,\Phi^2}} \quad \text{hence we deduce} \quad |Jac\,(\mathcal{G}_{el}^{-1})\,| \leqslant \xi\,\Phi^2.$$

Since Φ is bounded, there exists $\kappa > 0$ independant of ε such that $|Jac\,(\mathcal{G}_{el}^{-1})\,| \leqslant \kappa$. Moreover note that

$$\frac{\xi'^2 - \nu'^2}{\xi^2 - \nu^2} = \frac{1 + (\xi^2 - 1)\Phi^2 - \nu^2}{\xi^2 - \nu^2} = \frac{\xi^2 - 1}{\xi^2 - \nu^2}\Phi^2 + \frac{1 - \nu^2}{\xi^2 - \nu^2} \leqslant 1 + \Phi^2 \leqslant \kappa.$$

From this we obtain

$$\begin{aligned}
&\int_{\xi'=1}^{\xi'_*}\int_{-1}^{+1}\int_0^{2\pi} |u \circ \mathcal{G} \circ \phi_{el}^{-1}\,(\xi', \nu', \theta')|^2 (\xi'^2 - \nu'^2) d\xi' d\nu' d\theta' \\
&= \int_{\xi=1}^{\xi_*}\int_{-1}^{+1}\int_0^{2\pi} |u \circ \phi_{el}^{-1}\,(\xi, \nu, \theta)|^2 \frac{\xi'^2 - \nu'^2}{\xi^2 - \nu^2} |Jac\,(\mathcal{G}_{el}^{-1})\,| (\xi^2 - \nu^2) d\xi d\nu d\theta \\
&\leqslant \kappa \int_{\xi=1}^{\xi_*}\int_{-1}^{+1}\int_0^{2\pi} |u \circ \phi_{el}^{-1}\,(\xi, \nu, \theta)|^2\,(\xi^2 - \nu^2) d\xi d\nu d\theta
\end{aligned}$$

which implies

$$\|u \circ \mathcal{G}\|_{L^2(\Omega_R)} \leqslant \kappa\,\|u\|_{L^2(\Omega_R)}.$$

For the estimate on the L^2 norm of the gradient, here is some preliminary computation.

$$\begin{aligned}
\frac{\partial(u\circ\mathcal{G})}{\partial\xi'} &= \frac{\partial\xi}{\partial\xi'}\,(\frac{\partial u}{\partial\xi}\circ\mathcal{G}) && = \frac{\xi'}{\Phi^2\sqrt{1+\frac{\xi'^2-1}{\Phi^2}}}\,(\frac{\partial u}{\partial\xi}\circ\mathcal{G}),\\
\frac{\partial(u\circ\mathcal{G})}{\partial\nu'} &= \frac{\partial\xi}{\partial\nu'}\,(\frac{\partial u}{\partial\xi}\circ\mathcal{G})+\frac{\partial\nu}{\partial\nu'}\,(\frac{\partial u}{\partial\nu}\circ\mathcal{G}) && = -\frac{\partial\Phi}{\partial\nu'}\frac{\xi'^2-1}{\Phi^3\sqrt{1+\frac{\xi'^2-1}{\Phi^2}}}(\frac{\partial u}{\partial\xi}\circ\mathcal{G})+(\frac{\partial u}{\partial\nu}\circ\mathcal{G}),\\
\frac{\partial(u\circ\mathcal{G})}{\partial\theta'} &= \frac{\partial\xi}{\partial\theta'}\,(\frac{\partial u}{\partial\xi}\circ\mathcal{G})+\frac{\partial\theta}{\partial\theta'}\,(\frac{\partial u}{\partial\theta}\circ\mathcal{G}) && = -\frac{\partial\Phi}{\partial\theta'}\frac{\xi'^2-1}{\Phi^3\sqrt{1+\frac{\xi'^2-1}{\Phi^2}}}(\frac{\partial u}{\partial\xi}\circ\mathcal{G})+(\frac{\partial u}{\partial\theta}\circ\mathcal{G}).
\end{aligned}$$

Then we obtain the following estimates (the constants κ being independant of ε):

$$\begin{aligned}
(\xi'^2-1)|\frac{\partial(u\circ\mathcal{G})}{\partial\xi'}|^2 &\leqslant \kappa\,(\xi^2-1)|\frac{\partial u}{\partial\xi}|^2,\\
(1-\nu'^2)|\frac{\partial(u\circ\mathcal{G})}{\partial\nu'}|^2 &\leqslant \kappa\,(1-\nu'^2)(\xi^2-1)\,|\frac{\partial u}{\partial\xi}|^2+\kappa\,(1-\nu^2)\,|\frac{\partial u}{\partial\nu}|^2\\
&\leqslant \kappa\,(\xi^2-1)\,|\frac{\partial u}{\partial\xi}|^2+\kappa\,(1-\nu^2)\,|\frac{\partial u}{\partial\nu}|^2,\\
\frac{\xi'^2-\nu'^2}{(\xi'^2-1)(1-\nu'^2)}|\frac{\partial(u\circ\mathcal{G})}{\partial\theta'}|^2 &\leqslant \kappa\,\frac{\xi'^2-\nu'^2}{(\xi'^2-1)(1-\nu^2)}|\frac{\partial\Phi}{\partial\theta'}|^2(\xi^2-1)^2|\frac{\partial u}{\partial\xi}|^2\\
&\quad+\kappa\,\frac{\xi'^2-\nu'^2}{\xi^2-\nu^2}\,\frac{\xi^2-\nu^2}{(\xi^2-1)(1-\nu^2)}|\frac{\partial u}{\partial\theta}|^2\\
&\leqslant \kappa\,(\xi^2-1)|\frac{\partial u}{\partial\xi}|^2+\kappa\,\frac{\xi^2-\nu^2}{(\xi^2-1)(1-\nu^2)}|\frac{\partial u}{\partial\theta}|^2.
\end{aligned}$$

Using the preceding estimates (independant of ε), we obtain the desired inequality for the L^2 norm of the gradient

$$\begin{aligned}
&\int_{\xi'=1}^{\xi'_*}\int_{-1}^{+1}\int_0^{2\pi}|\nabla(u\circ\mathcal{G})(\xi',\nu',\theta')|^2(\xi'^2-\nu'^2)d\xi'd\nu'd\theta'\\
&\quad\leqslant \kappa\int_{\xi=1}^{\xi'_*}\int_{-1}^{+1}\int_0^{2\pi}|\nabla u(\xi,\nu,\theta)|^2(\xi^2-\nu^2)|Jac\,(\mathcal{G}_{\mathrm{el}}^{-1})\;|d\xi d\nu d\theta\\
&\quad\leqslant \kappa\int_{\xi=1}^{\xi'_*}\int_{-1}^{+1}\int_0^{2\pi}|\nabla u(\xi,\nu,\theta)|^2(\xi^2-\nu^2)d\xi d\nu d\theta
\end{aligned}$$

which implies

$$\|\nabla(u\circ\mathcal{G})\|_{L^2(\Omega_R)}\leqslant\kappa\,\|\nabla u\|_{L^2(\Omega_R)}.$$

We will now construct a lifting operator mapping $H^{1/2}(\breve{\Gamma}^\varepsilon)$ into $H^1(\Omega_R)$ satisfying the properties we anounced. Take $u\in H^{1/2}(\breve{\Gamma}^\varepsilon)$. Denote that (ν,θ) the coordinates on $\breve{\Gamma}^\varepsilon$. Moreover define $\widehat{u}(l,m)=\int_{\nu=-1}^{+1}\int_0^{2\pi}u(\nu,\theta)\widetilde{P}_l^m(\nu,\theta)e^{-im\theta}d\nu,d\theta$. According to the preceding remarks

$$\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}(1+l^2)^{1/2}|\widehat{u}(l,m)|^2<+\infty\quad\text{and}\quad u=\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\widehat{u}(l,m)\widetilde{P}_l^m(\nu)e^{im\theta}.$$

The convergence must be considered in the sense of $H^{1/2}(\breve{\Gamma}^\varepsilon)$. take a C^∞ cut-off function $\chi:\mathbb{R}\to[0;1]$ decreasing such that $\chi(x)=0$ for $x\geqslant1$ and $\chi(x)=1$ for $x\leqslant0$. We also introduce an auxiliary function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\alpha(r) = r$ if $0 \leqslant r \leqslant 1$ and $\alpha(r) = 1/r$ if $1 \leqslant r$. Note that $\alpha(1) = 1$. We define $\mathcal{R}^\varepsilon(u)$ in ellispoidal coordinates by

$$\mathcal{R}^\varepsilon(u)(\xi,\nu,\theta) = \chi(\xi)\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\widehat{u}(l,m)\,\widetilde{P}_l^m(\nu)e^{im\theta}\,\big(\alpha(\frac{\xi^2-1}{\varepsilon^2})\big)^{l/2} \tag{5}$$

Let us denote $\mathcal{R}^\varepsilon_{l,m}$ the terms of the sum above. Clearly this sum converges in the sense of $L^2(\Omega_R)$ since $|\alpha| \leqslant 1$ and $u \in L^2(\Gamma^\varepsilon)$. Now we verify that $\mathcal{R}^\varepsilon(u) \in H^1(\Omega_R)$ by proving that the sum (5) converges in $H^1(\Omega_R)$ and bound straightforwardly $\|\nabla\mathcal{R}^\varepsilon_{l,m}\|_{L^2(\Omega_R)}$. Note that the L^2 norm of the gradient of a function v over Ω_R is given by $\|\nabla v\|^2_{L^2(\Omega_R)} = \int_{\Omega_R}(\xi^2-1)|\frac{\partial v}{\partial\xi}|^2 + \frac{1}{\xi^2-1}|\frac{\partial v}{\partial\theta}|^2 d\xi\,d\nu\,d\theta + \int_{\Omega_R}(1-\nu^2)|\frac{\partial v}{\partial\nu}|^2 + \frac{1}{1-\nu^2}|\frac{\partial v}{\partial\theta}|^2 d\xi\,d\nu\,d\theta = T_\xi(v) + T_\nu(v)$. The norm $\|\nabla v\|^2_{L^2(\Omega_R)}$ thus splits in two terms that we successively estimate. Each term of the sum in (5) can be bounded as follows

$$\begin{aligned} T_\xi\big(\mathcal{R}^\varepsilon_{l,m}\big) &\leqslant |\widehat{u}(l,m)|^2\int_{\xi=1}^{\xi_*}(\xi^2-1)\Big|\frac{\partial}{\partial\xi}\,\big(\alpha(\frac{\xi^2-1}{\varepsilon^2})\big)^{l/2}\Big|^2 + \frac{l^2}{\xi^2-1}\big|\alpha(\frac{\xi^2-1}{\varepsilon^2})\big|^l\,d\xi \\ &\leqslant |\widehat{u}(l,m)|^2\int_{\xi=1}^{\xi_*} l^2\,\frac{\xi_*^2+1}{\xi^2-1}\,\big|\alpha(\frac{\xi^2-1}{\varepsilon^2})\big|^l\,d\xi. \end{aligned}$$

In the preceding estimate we have used the fact that $|r\alpha'(r)| = |\alpha(r)|$. We use the change of variable $\rho = \sqrt{\xi^2-1}/\varepsilon$ in order to bound the integral term for $l \geqslant 1$, and then find a constant $\kappa > 0$ independant of ε such that

$$\int_{\xi=1}^{\xi_*}\frac{1}{\xi^2-1}\,\big|\alpha(\frac{\xi^2-1}{\varepsilon^2})\big|^l\,d\xi = \int_{\rho=0}^{\sqrt{\xi_*^2-1}/\varepsilon}\frac{|\alpha(\rho^2)|^l}{\sqrt{1+\varepsilon^2\rho^2}}\frac{d\rho}{\rho} \leqslant \frac{\kappa}{l+1}. \tag{6}$$

As a result there exists another constant $\kappa > 0$ independant of ε such that $T_\xi\big(\mathcal{R}^\varepsilon_{l,m}\big) \leqslant \kappa\,(1+l^2)^{1/2}|\widehat{u}(l,m)|^2$ and, since $\sum_{l,m}(1+l^2)^{1/2}|\widehat{u}(l,m)|^2$ converges, so does $\sum_{l,m}T_\xi\big(\mathcal{R}^\varepsilon_{l,m}\big)$. In order to bound the terms $T_\nu\big(\mathcal{R}^\varepsilon_{l,m}\big)$ we first note that

$$T_\nu\big(\mathcal{R}^\varepsilon_{l,m}\big) \leqslant\; |\widehat{u}(l,m)|^2\,\|\nabla_{S^2}\widetilde{P}_l^m(\nu)e^{im\theta}\|^2_{L^2(S^2)}\,\int_{\xi=1}^{\xi_*}|\alpha(\frac{\xi^2-1}{\varepsilon^2})|^l\,d\xi.$$

Let us recall the reader of the well known property of spherical harmonics $\|\nabla_{S^2}\widetilde{P}_l^m(\nu)e^{im\theta}\|^2_{L^2(S^2)} = l(1+l)$. In order to bound the integral term for $l \geqslant 1$, we simply use (6) which yields the existence of $\kappa > 0$ independant of ε such that $T_\nu\big(\mathcal{R}^\varepsilon_{l,m}\big) \leqslant \kappa\,(1+l^2)^{1/2}|\widehat{u}(l,m)|^2$, so $\sum_{l,m}T_\nu\big(\mathcal{R}^\varepsilon_{l,m}\big)$ converges. Gathering the estimates we have obtained on $T_\xi\big(\mathcal{R}^\varepsilon_{l,m}\big)$ and $T_\nu\big(\mathcal{R}^\varepsilon_{l,m}\big)$ and summing over $l \in \mathbb{N}$ and $m \in [-l,l]$, we are led to the existence of $\kappa > 0$ independant of ε such that $\|\mathcal{R}(u)\|_{H^1(\Omega_R)} \leqslant \kappa\,\|u\|_{H^{1/2}(\check{\Gamma}^\varepsilon)}$. Now let us define a lifting $\mathcal{R}$ mapping Γ^ε into $H^1(\Omega_R)$ by taking for any $u \in H^{1/2}(\Gamma^\varepsilon)$, $\mathcal{R}(u) = [\mathcal{R}(u\circ(\mathcal{G}^{-1})_{|\check{\Gamma}^\varepsilon})]\circ\mathcal{G}$. Finally we conclude the lemma combining the two preceding results established in this proof,

$$\begin{aligned} \|\mathcal{R}(u)\|_{H^1(\Omega_R)} &= \|[\mathcal{R}(u\,\circ\,(\mathcal{G}^{-1})_{|\check{\Gamma}^\varepsilon})]\circ\mathcal{G}\|_{H^1(\Omega_R)} \leqslant \kappa\,\|\mathcal{R}(u\,\circ\,(\mathcal{G}^{-1})_{|\check{\Gamma}^\varepsilon})\|_{H^1(\Omega_R)} \\ &\leqslant \kappa\,\|u\,\circ\,(\mathcal{G}^{-1})_{|\check{\Gamma}^\varepsilon}\|_{H^{1/2}(\check{\Gamma}^\varepsilon)} = \kappa\,\|u\,\circ\,(\phi^{\Gamma^\varepsilon}_{\mathrm{el}})^{-1}\circ\phi^{\check{\Gamma}^\varepsilon}_{el}\|_{H^{1/2}(\check{\Gamma}^\varepsilon)} = \kappa\,\|u\|_{H^{1/2}(\Gamma^\varepsilon)}. \end{aligned}$$

■

2.4.4 Initial Problem

We now present precisely the problem we wish to study in the rest of this document. Consider $f \in L^2(\Omega_\varepsilon)$ with compact support supp$f = K$ satisfying $K\cap\overline{\Omega^i_\varepsilon} = \emptyset$ for ε small enough. We define

$u^\varepsilon \in H^2_{\mathrm{loc}}(\Omega_\varepsilon)$ as the unique solution of the problem

$$\begin{cases} \text{Find}\, u \in H^2_{\mathrm{loc}}(\Omega_\varepsilon)\, \text{such that} \\ \Delta u + k^2 u = f \quad \text{in} \quad \Omega_\varepsilon, \\ u = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \\ u \quad \text{outgoing.} \end{cases}$$

We can assume that u^ε is extended by 0 in Ω^i_ε, and then consider u^ε as an element of $\mathcal{H}$. If we denote n the outgoing normal with respect to Ω^i_ε and if we define

$$p^\varepsilon = \frac{\partial u^\varepsilon}{\partial n}|_{\Gamma^\varepsilon} \quad \text{where } n \text{ refer to the outgoing normal with respect to } \Omega^i_\varepsilon.$$

Then $(u^\varepsilon; p^\varepsilon)$ is the unique pair satisfying

$$(\mathcal{P}^\varepsilon) : \begin{cases} (u^\varepsilon; p^\varepsilon) \in \mathcal{H} \times H^{-1/2}(\Gamma^\varepsilon) \quad \text{such that} \\ \displaystyle\int_{\Omega_R} \nabla u^\varepsilon \cdot \nabla \overline{v} - k^2 \int_{\Omega_R} u^\varepsilon \overline{v} + \int_{\Gamma_R} \overline{v} T_R u^\varepsilon + \int_{\Gamma^\varepsilon} p^\varepsilon \overline{v} = -\int_{\Omega_R} f\overline{v} \quad \forall v \in \mathcal{H}, \\ \displaystyle\int_{\Gamma^\varepsilon} q u^\varepsilon = 0 \quad \forall q \in H^{-1/2}(\Gamma^\varepsilon). \end{cases}$$

This problem can reformulated in the following manner

$$(\mathcal{P}^\varepsilon) : \begin{cases} (u^\varepsilon; p^\varepsilon) \in \mathcal{H} \times H^{-1/2}(\Gamma^\varepsilon) \quad \text{such that} \\ (\mathcal{A}u^\varepsilon; v)_{\mathcal{H}} + \langle p^\varepsilon; v \rangle_{H^{-1/2}(\Gamma^\varepsilon),\, H^{1/2}(\Gamma^\varepsilon)} = -\displaystyle\int_{\Omega_R} f\overline{v} \quad \forall v \in \mathcal{H}, \\ \langle q; u^\varepsilon \rangle_{H^{-1/2}(\Gamma^\varepsilon),\, H^{1/2}(\Gamma^\varepsilon)} = 0 \quad \forall q \in H^{-1/2}(\Gamma^\varepsilon). \end{cases}$$

We also define $u_0 \in H^2_{loc}(\mathbb{R}^3)$ as the unique solution of the problem

$$\begin{cases} \text{Find}\, u \in H^2_{\mathrm{loc}}(\mathbb{R}^3)\, \text{such that} \\ \Delta u + k^2 u = f \quad \text{in} \quad \Omega, \\ u \quad \text{outgoing.} \end{cases}$$

3 Step 1: far field

In this section we will look for a function that is a good approximation of u^ε in the far field zone $\mathcal{Z}^\varepsilon_{\mathrm{lo}}$. Indeed we decided previously to represent the beginning of the expansion of u^ε in the form $u_0 + u^\varepsilon_1$. We have already defined u_0 in the preceding section. As a consequence we will propose in the next paragraph a problem being a caracterization of u^ε_1 as good as possible. Let us remind that our work ios for the moment only formal. We will be led to a problem that leaves only one thing undetermined: a function $a^\varepsilon \in L^2(I)$, the choice of which is delayed to a further section. In a third section we will study the behavior of u^ε_1 in the neighborhood of J.

3.1 Definition

Putting (1) in the Helmholtz equation verified by u^ε and using formal arguments:

$$\Delta u_0 + k^2 u_0 + \Delta u_1^\varepsilon + k^2 u_1^\varepsilon = f \quad \Rightarrow \left\{ \begin{array}{l} \Delta u_0 + k^2 u_0 = f, \\ \\ \Delta u_1^\varepsilon + k^2 u_1^\varepsilon = 0, \end{array} \right.$$

$$u_0 + u_1^\varepsilon \quad \text{outgoing} \quad \Rightarrow \left\{ \begin{array}{l} u_0 \quad \text{outgoing}, \\ \\ u_1^\varepsilon \quad \text{outgoing}. \end{array} \right.$$

We assumed that u_0 is the solution of a problem identical to $(\mathcal{P}^\varepsilon)$ except that there is no wire Γ^ε. Since the obstacle Ω_ε^i shrinks to J, we must assume that u_1^ε satisfies an homogeneous Helmholtz equation in $\mathbb{R}^3 \setminus J$. Since it satisfies an outgoing radiation condition, if u_1^ε is regular in the neighborhood of J it has to be 0, which is excluded. As a consequence u_1^ε admits a singularity in the neighborhood of J. We conjecture that u_1^ε admits only the weakest possible singularities (since it is the first term in the expansion of the diffracted field). We suppose that there exists $a^\varepsilon \in L^2(J)$ such that u_1^ε satisfies an equation of the following form

$$\left\{ \begin{array}{l} \Delta u_1^\varepsilon + k^2 u_1^\varepsilon = 0 \quad \text{in} \quad \Omega \setminus J, \\ \\ u_1^\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \\ \\ u_1^\varepsilon \quad \text{outgoing}, \\ \\ u_1^\varepsilon - \displaystyle\int_J a^\varepsilon(\mathbf{x}') \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\, d\mathbf{x}' \in H^1_{loc}(\Omega). \end{array} \right.$$

Denote $I =]-1;+1[$. Since $\mathbf{x}' \in J$ in the above formula, we will sometimes write I instead of J, and write z' instead of $\mathbf{x}'$.

3.2 Existence and uniqueness lemma

The following result will show that the equations we have written uniquely determine u_1^ε up to the choice of a^ε.

Lemma 3.1
Given $a \in L^2(J)$, the following problem admits a unique solution

$$\left\{ \begin{array}{l} \textit{Find } u \in H^1_{\mathrm{loc}}(\Omega \setminus J) \textit{ such that} \\ \\ \Delta u_1^\varepsilon + k^2 u_1^\varepsilon = 0 \quad \textit{in} \quad \Omega \setminus J, \\ \\ u_1^\varepsilon = 0 \quad \textit{on} \quad \partial\Omega, \\ \\ u_1^\varepsilon \quad \textit{outgoing}, \\ \\ u_1^\varepsilon - \displaystyle\int_J a(\mathbf{x}') \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\, d\mathbf{x}' \in H^1_{loc}(\Omega). \end{array} \right.$$

Proof:

Take a C^∞ cut-off function $\chi : \Omega \mapsto \mathbb{R}$ decreasing such that there exists $\xi_0 > 0$ like in §2.3.3 for which $\chi(\mathbf{x}) = 1$ if $\xi < \xi_0$ and $\chi(\mathbf{x}) = 0$ if $\xi > 2\xi_0$. Denote

$$\widetilde{u}(\mathbf{x}) = \chi(\mathbf{x}) \int_J a(\mathbf{x}') \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\, d\mathbf{x}'.$$

Denote $g = -\Delta\widetilde{u} - k^2\widetilde{u}$ in $\Omega \setminus J$ and $g = 0$ on J. Take $\widetilde{v} \in H^1_{\mathrm{loc}}(\Omega)$ uniquely satisfying:

$$\begin{cases} \Delta\widetilde{v} + k^2\widetilde{v} = g \quad \text{in} \quad \Omega, \\ \widetilde{v} = 0 \quad \text{on} \quad \partial\Omega, \\ \widetilde{v} \quad \text{outgoing.} \end{cases}$$

Defining $u = \widetilde{u} + \widetilde{v}$ we get a solution of the problem we are interested in, hence the existence result. For the uniqueness, if u and v are two solutions of this problem, then $u - v$ satisfies:

$$\begin{cases} u - v \in H^1_{loc}(\Omega) \\ \Delta(u-v) + k^2(u-v) = 0 \quad \text{in} \quad \Omega, \\ u - v = 0 \quad \text{on} \quad \partial\Omega, \\ u - v \quad \text{outgoing.} \end{cases}$$

hence the uniqueness result, by means of the uniqueness of the solution of the usual homogeneous Helmholtz problem.

■

Supposing that a^ε is known (we delay its choice to a further paragraph), we will now describe u_1^ε. Denote $G(\mathbf{x};\mathbf{x}') = G_{\mathbf{x}'}(\mathbf{x})$ the unique function satisfying, for any $\mathbf{x}' \in \Omega$,

$$\begin{cases} G_{\mathbf{x}'} \in H^1_{loc}(\Omega \setminus \{\mathbf{x}'\}) \\ \Delta G_{\mathbf{x}'} + k^2 G_{\mathbf{x}'} = \delta_{\mathbf{x}'} \quad \text{in} \quad \Omega \\ G_{\mathbf{x}'} = 0 \quad \text{on} \quad \partial\Omega \\ G_{\mathbf{x}'} \quad \text{outgoing} \end{cases}$$

For $\mathbf{x}' \in \Omega$, define $G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}') = G^{\mathrm{reg}}_{\mathbf{x}'}(\mathbf{x})$. The unique function satisfying

$$\begin{cases} G^{\mathrm{reg}}_{\mathbf{x}'} \in H^1_{loc}(\Omega), \\ \Delta G^{\mathrm{reg}}_{\mathbf{x}'} + k^2 G^{\mathrm{reg}}_{\mathbf{x}'} = 0 \quad \text{in} \quad \Omega, \\ G^{\mathrm{reg}}_{\mathbf{x}'}(\mathbf{x}) = -\dfrac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \quad \text{for} \quad \mathbf{x} \in \partial\Omega, \\ G^{\mathrm{reg}}_{\mathbf{x}'} \quad \text{outgoing.} \end{cases}$$

Basical computations show that $(\mathbf{x};\mathbf{x}') \mapsto G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}')$ is a C^∞ function in $\mathcal{O} \times \mathcal{O}$. It is also easy to see that

$$G(\mathbf{x};\mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} + G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}'). \tag{7}$$

For any $\mathbf{x}' \in J$, $G^{\mathrm{reg}}_{\mathbf{x}'}$ is a solution to the helmholtz equation and is smooth in the neighborhood of $\mathbf{x}'$. Using decomposition (7) we see that u_1^ε can be written

$$u_1^\varepsilon(\mathbf{x}) = \int_J a^\varepsilon(\mathbf{x}')\, G(\mathbf{x};\mathbf{x}')\, d\mathbf{x}' = \int_J a^\varepsilon(\mathbf{x}') \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\, d\mathbf{x}' + \int_J a^\varepsilon(\mathbf{x}')\, G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}')\, d\mathbf{x}'. \tag{8}$$

To conclude, we have defined a far field term u_1^ε. It is an harmonic acoustic field generated by a lineic set of source points distributed along J. Given an a^ε, u_1^ε is uniquely determined according to lemma 3.1.

3.3 Expansion in the neighborhood of J

3.3.1 Formal expansion

We have just seen that except for u_0, the terms of an expansion of u^ε have to admit a singular behavior in the neighborhood of J. Moreover u^ε is regular in the neighborhood of J. So we have to compensate this singular behavior: there appears a boundary layer around J. In order to construct the term related to this boundary layer, we have to know precisely the behavior of u_1^ε in the neighborhood of J i.e. when $\xi \to 1_+$. Thus we look for the expansion of u_1^ε when $\xi \to 1$. The computations for studying such an expansion will be formal in a first step. Using equations (8) and (21) of the appendix, we have

$$\begin{aligned} u_1^\varepsilon(\mathbf{x}) &= \int_J a^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}' + \int_J a^\varepsilon(\mathbf{x}')\,G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}')\,d\mathbf{x}' \\ &= \frac{a^\varepsilon(\mathbf{x})}{4\pi}\int_J \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} + \int_J \frac{a^\varepsilon(\mathbf{x}')-a^\varepsilon(\mathbf{x})}{4\pi|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}' + \int_J a^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}-1}{4\pi|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}' + \int_J a^\varepsilon(\mathbf{x}')\,G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}')\,d\mathbf{x}' \\ &\approx \underbrace{\frac{a^\varepsilon(\nu)}{4\pi}\ln\Big(\frac{2}{\xi-1}\Big) + \int_{-1}^{+1}\frac{a^\varepsilon(z)-a^\varepsilon(\nu)}{4\pi|z-\nu|}\,dz}_{\text{term 1}} + \underbrace{\int_{-1}^{+1} a^\varepsilon(z)\frac{e^{ik|\nu-z|}-1}{4\pi|\nu-z|}\,dz}_{\text{term 2}} + \underbrace{\int_{-1}^{+1} a^\varepsilon(z)\,G^{\mathrm{reg}}(\nu;z)\,dz}_{\text{term 3}} \\ &\quad + o(1) \end{aligned}$$

In the above calculus, $(\xi;\nu;\theta)$ refer to the ellipsoidal coordinates of $\mathbf{x}$ and $(1;z;\theta')$ refer to the ellipsoidal coordinates of $\mathbf{x}'$. We have used the following identity $|\mathbf{x}-\mathbf{x}'| = \sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}$ $\to$ $|\nu-z|$ when $\xi\to 1_+$. Denote

$$u_{1,as}^\varepsilon(\mathbf{x}) = \frac{a^\varepsilon(\nu)}{4\pi}\ln\Big(\frac{2}{\xi-1}\Big) + \int_{-1}^{+1}\frac{a^\varepsilon(z)-a^\varepsilon(\nu)}{4\pi|z-\nu|}\,dz + \int_{-1}^{+1} a^\varepsilon(z)\frac{e^{ik|\nu-z|}-1}{4\pi|\nu-z|}\,dz + \int_{-1}^{+1} a^\varepsilon(z)\,G^{\mathrm{reg}}(\nu;z)\,dz$$

3.3.2 Error estimate

We will now establish the validity of the above expansion by providing an estimate of the difference $u_1^\varepsilon - u_{1,as}^\varepsilon$ for two semi-norms

$$\|u_1^\varepsilon - u_{1,as}^\varepsilon\|^2_{L^\infty_\xi\otimes L^2_\nu(\mathcal{T}^\varepsilon)} = \sup_{\varepsilon<\xi^2-1<2\varepsilon}\int_{\nu=-1}^{+1}\int_{\theta=0}^{2\pi}|(u_1^\varepsilon - u_{1,as}^\varepsilon)(\xi;\nu;\theta)|^2\,d\theta\,d\nu,$$

$$\|\frac{\partial}{\partial\xi}\{u_1^\varepsilon - u_{1,as}^\varepsilon\}\|^2_{L^\infty_\xi\otimes L^2_\nu(\mathcal{T}^\varepsilon)} = \sup_{\varepsilon<\xi^2-1<2\varepsilon}\int_{\nu=-1}^{+1}\int_{\theta=0}^{2\pi}|\frac{\partial}{\partial\xi}(u_1^\varepsilon - u_{1,as}^\varepsilon)(\xi;\nu;\theta)|^2\,d\theta\,d\nu.$$

We will use decompositions in Legendre polynomials that introduce Legendre functions of first and second kind P_l and Q_l, and also functions denoted W_l. We refer the reader to the appendix A.2 for defintions and properties of these functions. Note that we will also use the result of lemma A.1 and corollary A.1 established in the appendix. Here $\widetilde{P}_l$ is proportional to the Legendre polynomial of order l, $\widetilde{P}_l = CP_l$ where C is chosen such that $\|\widetilde{P}_l\|_{L^2(I)} = 1$. Moreover in the computations that follow, we assume that $\varepsilon < \xi^2-1 < 2\varepsilon$.

- **Term 1**

$$\int_{\nu=-1}^{+1}\Big|\int_J \frac{a^\varepsilon(\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}' - \frac{a^\varepsilon(\nu)}{4\pi}\ln\Big(\frac{2}{\xi-1}\Big) - \int_{-1}^{+1}\frac{a^\varepsilon(z)-a^\varepsilon(\nu)}{4\pi|z-\nu|}\,dz\Big|^2 d\nu =$$

$$\int_{\nu=-1}^{+1}\Big|\frac{1}{2\pi}\sum_{l=0}^{+\infty}\Big\langle a^\varepsilon;\widetilde{P}_l\Big\rangle_{L^2(I)}\widetilde{P}_l(\nu)\Big\{Q_l(\xi)-\frac{1}{2}\ln\big(\frac{2}{\xi-1}\big)-W_{l-1}(1)\Big\}\Big|^2 d\nu =$$

$$\frac{1}{4\pi^2}\sum_{l=0}^{+\infty}\Big|\Big\langle a^\varepsilon;\widetilde{P}_l\Big\rangle_{L^2(I)}\Big|^2\,\Big|Q_l(\xi)-\frac{1}{2}\ln\big(\frac{2}{\xi-1}\big)-W_{l-1}(1)\Big|^2 \leqslant$$

$$\frac{\kappa}{4\pi^2}\sum_{l=0}^{+\infty} l(l+1)\ln^2 l\,\Big|\Big\langle a^\varepsilon;\widetilde{P}_l\Big\rangle_{L^2(I)}\Big|^2\,(\xi^2-1)\ln\frac{1}{\xi^2-1}$$

We obtain from the above calculus

$$\Big\|\int_J \frac{a^\varepsilon(\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}' - \frac{a^\varepsilon(\nu)}{4\pi}\ln\Big(\frac{\xi+1}{\xi-1}\Big) - \int_{-1}^{+1}\frac{a^\varepsilon(z)-a^\varepsilon(\nu)}{4\pi|z-\nu|}\,dz\Big\|_{L^\infty_\xi\otimes L^2_\nu(\mathcal{T}^\varepsilon)} \leqslant \kappa\,\|a^\varepsilon\|_{E^2(I)}\,\varepsilon^{1/2}\ln 1/\varepsilon$$

The same computation can be derived for the derivative with respect to ξ using corollaries A.5 and A.6,

$$\int_{\nu=-1}^{+1}\Big|\frac{\partial}{\partial\xi}\int_J \frac{a^\varepsilon(\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}' - \frac{\partial}{\partial\xi}\frac{a^\varepsilon(\nu)}{4\pi}\ln\Big(\frac{2}{\xi-1}\Big)\Big|^2 d\nu =$$

$$\int_{\nu=-1}^{+1}\Big|\frac{1}{2\pi}\sum_{l=0}^{+\infty}\Big\langle a^\varepsilon;\widetilde{P}_l\Big\rangle_{L^2(I)}\widetilde{P}_l(\nu)\frac{\partial}{\partial\xi}\Big\{Q_l(\xi)-\frac{1}{2}\ln\big(\frac{2}{\xi-1}\big)\Big\}\Big|^2 d\nu =$$

$$\frac{1}{4\pi^2}\sum_{l=0}^{+\infty}\Big|\Big\langle a^\varepsilon;\widetilde{P}_l\Big\rangle_{L^2(I)}\Big|^2\,\Big|\frac{\partial}{\partial\xi}\Big\{Q_l(\xi)-\frac{1}{2}\ln\big(\frac{2}{\xi-1}\big)\Big\}\Big|^2 \leqslant$$

$$\frac{\kappa}{4\pi^2}\sum_{l=0}^{+\infty} l(l+1)\ln^2 l\,\Big|\Big\langle a^\varepsilon;\widetilde{P}_l\Big\rangle_{L^2(I)}\Big|^2\,\frac{1}{\xi^2-1}\,\ln\frac{1}{\xi^2-1}.$$

We obtain from the above calculus

$$\Big\|\frac{\partial}{\partial\xi}\Big\{\int_J \frac{a^\varepsilon(\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}' - \frac{a^\varepsilon(\nu)}{4\pi}\ln\Big(\frac{\xi+1}{\xi-1}\Big) - \int_{-1}^{+1}\frac{a^\varepsilon(z)-a^\varepsilon(\nu)}{4\pi|z-\nu|}\,dz\Big\}\Big\|_{L^\infty_\xi\otimes L^2_\nu(\mathcal{T}^\varepsilon)}$$
$$\leqslant \kappa\,\|a^\varepsilon\|_{E^2(I)}\,\frac{1}{\varepsilon^{1/2}}\ln 1/\varepsilon$$

- **Term 2**

If $\mathbf{x}$ is a point with ellipsoidal coordinates $(\xi;\nu;\theta)$, define $\mathbf{x}_J$ as the point with ellipsoidal coordinates $(1;\nu;\theta)$. Then

$$\int_{-1}^{+1} a^\varepsilon(z)\frac{e^{ik|\nu-z|}-1}{4\pi|\nu-z|}\,dz = \int_J a^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}_J-\mathbf{x}'|}-1}{4\pi|\mathbf{x}_J-\mathbf{x}'|}\,d\mathbf{x}'$$

$$\int_J a^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}-1}{4\pi|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}' - \int_J a^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}_J-\mathbf{x}'|}-1}{4\pi|\mathbf{x}_J-\mathbf{x}'|}\,d\mathbf{x}' =$$

$$\int_J\int_0^1 a^\varepsilon(\mathbf{x}')\{e^{itk|\mathbf{x}-\mathbf{x}'|}-e^{itk|\mathbf{x}_J-\mathbf{x}'|}\}\,dt\,d\mathbf{x}'.$$

Using an order 1 Taylor equality, we obtain:

$$|e^{itk|\mathbf{x}-\mathbf{x}'|} - e^{itk|\mathbf{x}_J-\mathbf{x}'|}| \leqslant t\,k\,\Big||\mathbf{x}-\mathbf{x}'| - |\mathbf{x}_J-\mathbf{x}'|\Big| \leqslant t\,k\,|\mathbf{x}-\mathbf{x}_J|,$$

hence the inequality

$$\Big\|\int_J a^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}-1}{4\pi|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}' - \int_J a^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}_J-\mathbf{x}'|}-1}{4\pi|\mathbf{x}_J-\mathbf{x}'|}\,d\mathbf{x}'\Big\|_{L^\infty_\xi\otimes L^2_\nu(\mathcal{T}^\varepsilon)} \leqslant \kappa\,\|a^\varepsilon\|_{E^0(I)}\,\varepsilon^{1/2}.$$

For the estimate in the second semi-norm

$$\frac{\partial}{\partial\xi}\int_J a^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}-1}{4\pi|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}' = \int_J\int_0^1 a^\varepsilon(\mathbf{x}')\,itk\,\frac{\partial}{\partial\xi}\{|\mathbf{x}-\mathbf{x}'|\}\,e^{itk|\mathbf{x}-\mathbf{x}'|}\,dt\,d\mathbf{x}'.$$

Here is an auxiliary computation ($\mathbf{x}(\xi;\nu;\theta)$ and $\mathbf{x}'(0;z;\theta)$):

$$\frac{\partial}{\partial\xi}\{|\mathbf{x}-\mathbf{x}'|\} = \frac{\partial}{\partial\xi}\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2} = \frac{\xi-\nu z}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}}.$$

Note that

$$(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2 = (\xi-1)(\xi-\nu z) + \underbrace{(\xi-1)(1-\nu z)+(\nu-z)^2}_{\geqslant 0}$$

Gathering the above results we obtain

$$\Big\|\frac{\partial}{\partial\xi}\Big\{\int_J a^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}-1}{4\pi|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}' - \int_J a^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}_J-\mathbf{x}'|}-1}{4\pi|\mathbf{x}_J-\mathbf{x}'|}\,d\mathbf{x}'\Big\}\Big\|_{L^\infty_\xi\otimes L^2_\nu(\mathcal{T}^\varepsilon)} \leqslant \kappa\,\|a^\varepsilon\|_{E^0(I)}\,\frac{1}{\varepsilon^{1/2}}.$$

- **Term 3**

Using the notations of the preceding section, we have

$$\int_J a^\varepsilon(\mathbf{x}')\,G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}')\,d\mathbf{x}' - \int_J a^\varepsilon(\mathbf{x}')\,G^{\mathrm{reg}}(\mathbf{x}_J;\mathbf{x}')\,d\mathbf{x}' = \int_J\int_0^1 a^\varepsilon(\mathbf{x}')\,\nabla G^{\mathrm{reg}}_{\mathbf{x}'}\cdot(\mathbf{x}-\mathbf{x}_J)\,d\mathbf{x}'.$$

Keeping in mind the smoothness of G^{reg} this yields

$$\Big\|\int_J a^\varepsilon(\mathbf{x}')\,G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}')\,d\mathbf{x}' - \int_J a^\varepsilon(\mathbf{x}')\,G^{\mathrm{reg}}(\mathbf{x}_J;\mathbf{x}')\,d\mathbf{x}'\Big\|_{L^\infty_\xi\otimes L^2_\nu(\mathcal{T}^\varepsilon)} \leqslant \kappa\,\|a^\varepsilon\|_{E^0(I)}\,\varepsilon^{1/2}.$$

For the estimate in the second semi-norm

$$\begin{aligned}\Big|\frac{\partial}{\partial\xi}\int_J a^\varepsilon(\mathbf{x}')\,G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}')\,d\mathbf{x}'\Big|^2 &= \Big(\frac{\xi^2-\nu^2}{\xi^2-1}\Big)\Big|\int_J a^\varepsilon(\mathbf{x}')\Big(\frac{\xi^2-1}{\xi^2-\nu^2}\Big)^{1/2}\frac{\partial}{\partial\xi}G^{\mathrm{reg}}_{\mathbf{x}'}(\mathbf{x})\,d\mathbf{x}'\Big|^2\\ &\leqslant \Big(\frac{\xi^2-\nu^2}{\xi^2-1}\Big)\int_J|a^\varepsilon(\mathbf{x}')|^2 d\mathbf{x}'\int_J\underbrace{\Big(\frac{\xi^2-1}{\xi^2-\nu^2}\Big)\Big|\frac{\partial}{\partial\xi}G^{\mathrm{reg}}_{\mathbf{x}'}(\mathbf{x})\Big|^2}_{\leqslant|\nabla G^{\mathrm{reg}}_{\mathbf{x}'}|^2}\,d\mathbf{x}'.\end{aligned}$$

Again the smoothness of G^{reg} provides

$$\|\frac{\partial}{\partial\xi}\Big\{\int_J a^\varepsilon(\mathbf{x}')\,G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}')\,d\mathbf{x}' - \int_J a^\varepsilon(\mathbf{x}')\,G^{\mathrm{reg}}(\mathbf{x}_J;\mathbf{x}')\,d\mathbf{x}'\Big\}\|_{L^\infty_\xi\otimes L^2_\nu(\mathcal{T}^\varepsilon)} \leqslant \kappa\,\|a^\varepsilon\|_{E^0(I)}\,\frac{1}{\varepsilon^{1/2}}.$$

- **Conclusion**

To conclude we have obtain the following estimates

$$\|u_1^\varepsilon - u_{1,as}^\varepsilon\|^2_{L^\infty_\xi\otimes L^2_\nu(\mathcal{T}^\varepsilon)} \leqslant \kappa\,\|a^\varepsilon\|_{E^3(I)}\,\varepsilon^{1/2}\,\ln 1/\varepsilon,$$

$$\|\frac{\partial}{\partial\xi}\{u_1^\varepsilon - u_{1,as}^\varepsilon\}\|^2_{L^\infty_{\xi,1}\otimes L^2_\nu(\mathcal{T}^\varepsilon)} \leqslant \kappa\,\|a^\varepsilon\|_{E^3(I)}\,\frac{1}{\varepsilon^{1/2}}\,\ln 1/\varepsilon.$$

4 Step 2: near field

In this section we will look for a good approximation of u^ε in the near field zone $\mathcal{Z}^\varepsilon_{\mathrm{pr}}$. However, since this zone varies with ε, we will use a suitable change of coordinates: we will use scaled coordinates. In a first paragraph we will formulate several remarks concerning the new geometrical setting that we are led to consider. The second paragraph will dedicated to the formal search for a problem that would caracterize at best U^ε_1 i.e. the predominant behavior of u^ε in the near field zone. We will be led to a problem that caracterize U^ε_1 up to the choice of a one dimensional function $b^\varepsilon \in L^2(I)$ that we will choose in a further section. In the third paragraph we will study the behavior of U^ε_1 in the neighborhood of infinity.

4.1 Change of coordinate

We recall the equation defining the surface Γ^ε in the scaled coordinates,

$$(\Gamma^\varepsilon) : \zeta = \Phi(\nu;\theta).$$

For a better interpretation of future results we introduce two new geometrical spaces.

4.1.1 Normalized geometrical space

The first of these spaces is denoted $\widehat{\mathcal{E}}$. We denote $\widehat{\mathbf{x}}$ the points of $\widehat{\mathcal{E}}$. Cartesian coordinates of a point $\widehat{\mathbf{x}}$ will be denoted $(\widehat{x};\widehat{y};\widehat{z})$ and cylindrical coordinates will be denoted $(\zeta;\theta;\nu)$, so

$$\widehat{x} = \zeta\cos\theta \qquad \widehat{y} = \zeta\sin\theta \qquad \widehat{z} = \nu$$

We also define $\Omega_N = \{\widehat{\mathbf{x}}(\zeta;\theta;\nu) \in \widehat{\mathcal{E}} \mid \zeta > \Phi(\nu;\theta)\}$. We can exhibit an explicit relation between Ω_ε and Ω_N, $\mathbf{x}(\zeta;\nu;\theta) \in \Omega_\varepsilon \;\;\Leftrightarrow\;\; \widehat{\mathbf{x}}(\zeta;\theta;\nu) \in \Omega_N$. Note that $\widehat{\mathcal{E}}$ is another geometrical space put in correspondace with $\mathcal{E}$. In $\widehat{\mathcal{E}}$ the wire does not depend on ε anymore. Looking at problems in $\widehat{\mathcal{E}}$ instead of $\mathcal{E}$ is like using a magnifying glass.

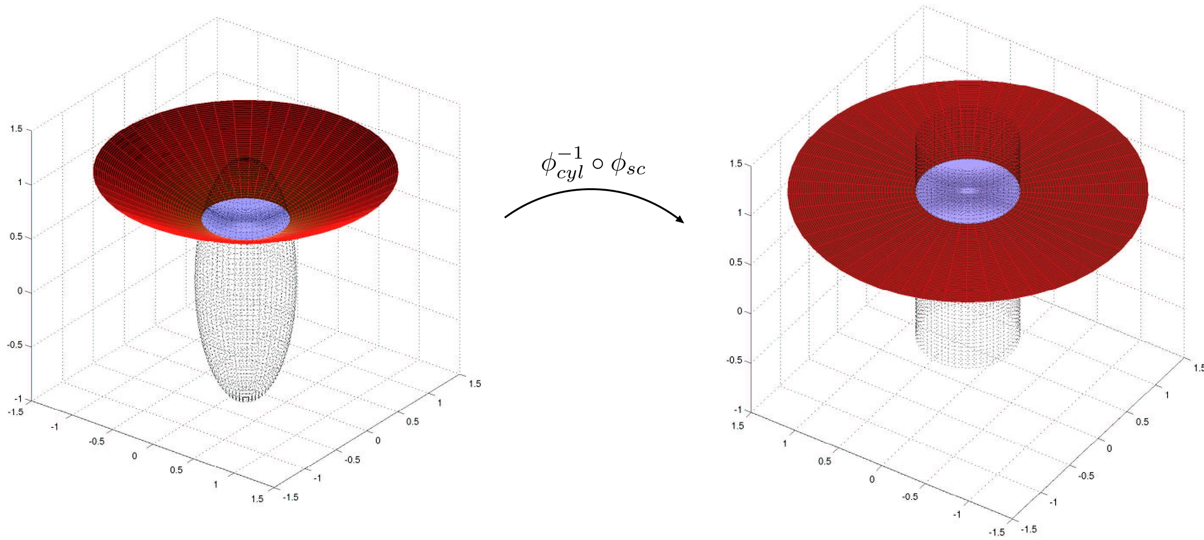
4.1.2 Transverse geometrical space

The second of these spaces is a geometrical plane that we denote $\widehat{\mathcal{E}}_\perp$. We will denote $\widehat{\mathbf{x}}_\perp$ the points of $\widehat{\mathcal{E}}_\perp$, and their polar coordinates will be denoted $(\zeta;\theta)$. We will identify a point of $(\widehat{\mathbf{x}}_\perp(\widehat{x};\widehat{y});\widehat{z}) \in \widehat{\mathcal{E}}_\perp \times \mathbb{R}$ with the point $\widehat{\mathbf{x}}(\widehat{x};\widehat{y};\widehat{z}) \in \widehat{\mathcal{E}}$ so $\widehat{\mathcal{E}}_\perp \times \mathbb{R}$ corresponds to decomposition of $\widehat{\mathcal{E}}$ in "slices". We will also denote:

$$\omega_N(\nu) = \{\widehat{\mathbf{x}}_\perp(\zeta;\theta) \in \widehat{\mathcal{E}}_\perp \mid \zeta > \Phi(\nu;\theta)\}$$

$$\omega^i_N(\nu) = \widehat{\mathcal{E}}_\perp \setminus \overline{\omega_N(\nu)}$$

In the figure below we have represented in red the transverse space for a given ν in a very simple geometrical setting.



4.2 Formal calculus and definition

We first make some simple computational remarks.

$$\xi^2 - 1 = \varepsilon^2 \zeta^2 \quad , \quad \frac{\partial \xi}{\partial \zeta} = \varepsilon^2 \frac{\zeta}{\xi} \quad \text{et} \quad \frac{\partial}{\partial \xi} = \frac{\xi}{\varepsilon^2 \zeta} \frac{\partial}{\partial \zeta}$$

The Helmholtz equation verified by u^εcan be written in scaled coordinates in the following way

$$\Delta u^\varepsilon + k^2 u^\varepsilon = 0$$

$$\frac{\partial}{\partial \xi}(\xi^2 - 1)\frac{\partial u^\varepsilon}{\partial \xi} + k^2(\xi^2 - 1)u^\varepsilon + \frac{1}{\xi^2 - 1}\frac{\partial^2 u^\varepsilon}{\partial \theta^2} + \frac{\partial}{\partial \nu}(1 - \nu^2)\frac{\partial u^\varepsilon}{\partial \nu} + k^2(1 - \nu^2)u^\varepsilon + \frac{1}{1 - \nu^2}\frac{\partial^2 u^\varepsilon}{\partial \theta^2} = 0$$

$$\frac{\xi}{\varepsilon^2 \zeta}\frac{\partial}{\partial \zeta}\varepsilon^2 \zeta^2 \frac{\xi}{\varepsilon^2 \zeta}\frac{\partial u^\varepsilon}{\partial \zeta} + k^2 \varepsilon^2 \zeta^2 u^\varepsilon + \frac{1}{\varepsilon^2 \zeta^2}\frac{\partial^2 u^\varepsilon}{\partial \theta^2} + \frac{\partial}{\partial \nu}(1 - \nu^2)\frac{\partial u^\varepsilon}{\partial \nu} + k^2(1 - \nu^2)u^\varepsilon + \frac{1}{1 - \nu^2}\frac{\partial^2 u^\varepsilon}{\partial \theta^2} = 0$$

$$\frac{1}{\varepsilon^2}\Big(\frac{1}{\zeta}\frac{\partial}{\partial \zeta}\zeta \frac{\partial u^\varepsilon}{\partial \zeta} + \frac{1}{\zeta^2}\frac{\partial^2 u^\varepsilon}{\partial \theta^2}\Big) + \zeta \frac{\partial}{\partial \zeta}\zeta \frac{\partial u^\varepsilon}{\partial \zeta} + \zeta \frac{\partial u^\varepsilon}{\partial \zeta}$$

$$+\frac{\partial}{\partial \nu}(1 - \nu^2)\frac{\partial u^\varepsilon}{\partial \nu} + k^2(1 - \nu^2)u^\varepsilon + \frac{1}{1 - \nu^2}\frac{\partial^2 u^\varepsilon}{\partial \theta^2} + k^2 \varepsilon^2 \zeta^2 u^\varepsilon = 0.$$

A good approximation of u^ε will cancel the dominant term with respect to ε in the above equation. Here it looks like a transverse Laplace operator. The near field denoted U_1^ε will have to verify in scaled coordinates $\frac{1}{\zeta}\frac{\partial}{\partial \zeta}\zeta \frac{\partial U_1^\varepsilon}{\partial \zeta} + \frac{1}{\zeta^2}\frac{\partial^2 U_1^\varepsilon}{\partial \theta^2} = 0$. Thus, instead of caracterizing $U_1^\varepsilon(\cdot)$ ina single time, we will caracterize $U_1^\varepsilon(\cdot;\nu)$ for each ν. Let us define $\widehat{U}_1^\varepsilon$ and $\widehat{U}_{1,\nu}^\varepsilon$ by

$$\widehat{U}_1^\varepsilon(\widehat{\mathbf{x}}) = \widehat{U}_{1,\nu}^\varepsilon(\widehat{\mathbf{x}}_\perp) = U_1^\varepsilon(\mathbf{x}).$$

Denoting $\Delta_\perp$ the transverse Laplace operator for functions on $\widehat{\mathcal{E}}_\perp$, we are led to a Lpalace equation for each ν,

$$\forall \nu \in [-1;+1], \qquad \Delta_\perp \widehat{U}^\varepsilon_{1,\nu} = 0 \quad \text{in} \quad \omega_N(\nu).$$

Moreover, to caracterize $\widehat{U}^\varepsilon_{1,\nu}$ we have to impose boundary conditions. Since for any ε, $u^\varepsilon = 0$ on Γ^ε we are led to impose $\widehat{U}^\varepsilon_{1,\nu} = 0$ on $\partial\omega_N(\nu)$. Finally we have to impose a condition at infinity. If we imposed $\widehat{U}^\varepsilon_{1,\nu}$ to be bounded at infinity, then necessarily $U^\varepsilon_{1,\nu} = 0$, which is excluded because we have to compensate the singularity of the far field u^ε_1 in the neighborhood of J. So we sill impose a growing behavior at infinity, and since the singularity of u^ε_1 is logarithmic, we will impose a logarithmic behavior. To sum up $\forall \nu \in [-1;+1]$:

$$\begin{cases} \Delta_\perp \widehat{U}^\varepsilon_{1,\nu} = 0 \quad \text{in} \quad \omega_N(\nu), \\ \widehat{U}^\varepsilon_{1,\nu} = 0 \quad \text{on} \quad \partial\omega_N(\nu), \\ \widehat{U}^\varepsilon_{1,\nu} \quad \text{admits a logarithmic behavior at infinity.} \end{cases}$$

However the previous problem only caracterizes $\widehat{U}^\varepsilon_{1,\nu}$ up to an additive constant. Indeed, we state result of existence and uniqueness for the solutions of Laplace problems with logarithmic condition at infinity. Take $R_N > 0$ such that $\forall \nu \in [-1;+1]$, $\omega^i_N(\nu) \subset D_{R_N}$ (the disk of center 0 with radius R_N is denoted D_{R_N}). Let us introduce

$$W^{1,-1}(\widehat{\mathcal{E}}_\perp \backslash \overline{D_{R_N}}) = \{U \in L^2_{loc}(\widehat{\mathcal{E}}_\perp \backslash \overline{D_{R_N}}) \quad | \quad \frac{U(\widehat{\mathbf{x}}_\perp)}{|\widehat{\mathbf{x}}_\perp| \ln |\widehat{\mathbf{x}}_\perp|} \in L^2(\widehat{\mathcal{E}}_\perp \backslash \overline{D_{R_N}}) \quad \text{and} \quad \nabla U \in L^2(\widehat{\mathcal{E}}_\perp \backslash \overline{D_{R_N}})\}.$$

which is, according to [15], the natural variational space for bidimensional Laplace problems in unbounded domains. In particular a function $U \in W^{1,-1}(\widehat{\mathcal{E}}_\perp \setminus \overline{D_{R_N}})$ is bounded at infinity if and only if $\limsup\limits_{\zeta \to +\infty} \sup\limits_{\theta \in [0;2\pi]} |U(\zeta;\theta)| < +\infty$.

Lemma 4.1
For any $\nu \in [-1;+1]$, the problem:

$$\begin{cases} \textit{Find} \quad (U;\alpha) \in H^1_{loc}\big(\omega_N(\nu)\big) \times \mathbb{C} \quad \textit{such that} \\ \Delta U = 0 \quad \textit{in} \quad \omega_N(\nu), \\ U = 0 \quad \textit{on} \quad \partial\omega_N(\nu), \\ \lim\limits_{\zeta \to +\infty} \| U(\zeta;\theta) - \alpha - \ln \zeta \|_{\theta,\infty} = 0. \end{cases}$$

admits a unique solution.

Proof:
Suppose ν is fixed. For the existence, let $\widetilde{U}$ be the unique function in $W^{1,-1}(\omega_N(\nu))$ satisfying:

$$\Delta_\perp \widetilde{U} = 0 \quad \text{in} \quad \omega_N(\nu),$$

$$\widetilde{U}(\widehat{\mathbf{x}}_\perp) = -\ln|\widehat{\mathbf{x}}_\perp| \quad \text{for} \quad \widehat{\mathbf{x}}_\perp \in \partial\omega_N(\nu).$$

Then $\ln|\widehat{\mathbf{x}}_\perp| + \widetilde{U}(\widehat{\mathbf{x}}_\perp)$ is solution to the above problem. For the uniqueness, if V is another solution to this problem, then $U - V$ satisfies

$$\begin{cases} U - V \in W^{1,-1}(\omega_N(\nu)), \\ \Delta_\perp(U - V) = 0 \quad \text{in} \quad \omega_N(\nu), \\ U - V = 0 \quad \text{on} \quad \partial\omega_N(\nu). \end{cases}$$

hence the uniqueness result for our problem according to the uniqueness result for Laplace problem with boundedness at infinity (c.f [15] for exemple).

■

In the litterature the constant α appearing in the preceding lemma is called the capacity. Let us call $\widehat{V}_\nu$ the unique solution to the problem appearing in the statement of lemma 4.1. According to uniqueness, there exists $b^\varepsilon_\nu \in \mathbb{C}$ such that $\widehat{U}^\varepsilon_{1,\nu} = b^\varepsilon_\nu \widehat{V}_\nu$ To fully caracterize $U^\varepsilon_{1,\nu}$ there only remains to caracterize b^ε_ν. Defining the functions $b^\varepsilon(\nu) = b^\varepsilon(\mathbf{x}) = b^\varepsilon_\nu$, we have finally the equality

$$U^\varepsilon_1(\mathbf{x}) = b^\varepsilon(\mathbf{x})\, V(\mathbf{x}). \tag{9}$$

We see that the near field is constructed on the basis of a static problem i.e. the Laplace equation satisfied by V. Here is some justification of one of the assumptions of the Holland model that states that the field admits a static behavior in the neighborhood of the wire. Now we will state two properties, one related to $\widehat{V}_\nu$ and the other related to V. These properties will allow in section 8.1 the definition of an operator with "good properties" providing the average value of a function on each transverse section of the wire.

Lemma 4.2

For a fixed $\nu \in [-1,1]$, denote $\widehat{n}_\nu$ the outgoing normal vector to $\partial\omega_N(\nu)$ and $\widehat{\gamma}_\nu$ the surface measure on $\partial\omega_N(\nu)$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \widehat{V}_\nu}{\partial \widehat{n}_\nu}(\nu,\theta)\, \widehat{\gamma}_\nu(\theta)\, d\theta \;=\; 1.$$

Proof:

Since $\Delta_\perp \widehat{V}_\nu = 0$ let us apply a Green formula with the test function 1 in the domain $\omega_N(\nu)\cap D_r$ with $r \to +\infty$. We have

$$\int_0^{2\pi} \frac{\partial \widehat{V}_\nu}{\partial \widehat{n}_\nu}(\nu,\theta)\, \widehat{\gamma}_\nu(\theta)\, d\theta \;=\; \int_{\theta=0}^{2\pi} \frac{\partial \widehat{V}_\nu}{\partial r}\, r d\theta.$$

according to the expansion of $\widehat{V}_\nu$ in the neighborhood of infinity, $\frac{\partial \widehat{V}_\nu}{\partial r} \sim 1/r$ when $r \to +\infty$. Going to the limit for $r \to \infty$ in the above identity yields the desired result since the right hand side goes to 2π.

■

The preceding property has been deduced from a bidimensional context. We propose a three dimensional result similar to this last result. The difficulty comes from the longitudinal curvature (with respect to ν) of ellipsoidal coordinates.

Lemma 4.3

There exists $\kappa > 0$ independant of ε and ν such that for ε small enough and $\forall \nu \in I$

$$\Big|\frac{1}{2\pi}\int_0^{2\pi} \gamma^\varepsilon(\nu,\theta)\frac{\partial V}{\partial n}(\nu,\theta)\, d\theta - 1\Big| < \kappa\,\varepsilon.$$

Proof:

For $\mathcal{E}$, in scaled coordinates,we recall the expression of the metric tensor

$$d^2 m \;=\; \varepsilon^2 \frac{\xi^2-\nu^2}{\xi^2}\, d^2\zeta \;+\; \frac{\xi^2-\nu^2}{1-\nu^2}\, d^2\nu \;+\; \varepsilon^2\zeta^2(1-\nu^2)\, d^2\theta \;=\; h_1^2\, d^2\zeta \;+\; h_2^2\, d^2\nu \;+\; h_3^2\, d^2\theta.$$

The equation of Γ^ε with these coordinates is $(\Gamma^\varepsilon) : \zeta = \Phi(\nu,\theta)$. Denote $n = n_\zeta \frac{\partial}{\partial\zeta} + n_\nu \frac{\partial}{\partial\nu} + n_\theta \frac{\partial}{\partial\theta}$ the outgoing normal to Γ^ε. We can apply the result of lemma A.9 in the appendix and obtain

$$\gamma^\varepsilon n_\zeta = \frac{h_\nu h_\theta}{h_\zeta} \;=\; \Phi\,(1+\varepsilon^2\Phi^2)^{1/2} \;=\; \Phi + \frac{\varepsilon^2\Phi^3}{1+\sqrt{1+\varepsilon^2\Phi^2}}.$$

Moreover for $\widehat{\mathcal{E}}$, in scaled coordinates (ζ, θ) (polar coordinates), one easily computes the expression of the metric $\widehat{\gamma}_\nu$ on $\partial\omega_N(\nu)$, and of the outgoing normal $\widehat{n}_\nu = \widehat{n}_{\nu,\zeta}\frac{\partial}{\partial\zeta} + \widehat{n}_{\nu,\theta}\frac{\partial}{\partial\theta}$ to $\partial\omega_N(\nu)$

$$\widehat{\gamma}_\nu(\theta) = \Big(\Phi^2 + (\frac{\partial\Phi}{\partial\theta})^2\Big)^{1/2} \qquad \widehat{n}_{\nu,\zeta} = \frac{\Phi}{\Big(\Phi^2 + (\frac{\partial\Phi}{\partial\theta})^2\Big)^{1/2}} \qquad \widehat{\gamma}_\nu\, \widehat{n}_{\nu,\zeta} = \Phi.$$

From this we can deduce $\exists\kappa > 0$ such that $\forall\nu \in I, \forall\theta \in [0, 2\pi]$ $|\gamma^\varepsilon n_\zeta - \widehat{\gamma}_\nu\, \widehat{n}_{\nu,\zeta}| < \kappa\,\varepsilon^2$. Using the regularity of V, we can also establish the following result

$$\exists\kappa > 0 \quad \text{such that} \quad \forall\nu \in I, \forall\theta \in [0, 2\pi] \quad |\frac{n_\theta}{n_\zeta}\frac{\partial V}{\partial\theta} + \frac{n_\nu}{n_\zeta}\frac{\partial V}{\partial\nu} - \frac{\widehat{n}_{\nu,\theta}}{\widehat{n}_{\nu,\zeta}}\frac{\partial\widehat{V}_\nu}{\partial\theta}| < \kappa\,\varepsilon$$

Finally note that

$$\gamma^\varepsilon\frac{\partial V}{\partial n} = \gamma^\varepsilon n_\zeta\Big(\frac{\partial V}{\partial\zeta} + \frac{n_\theta}{n_\zeta}\frac{\partial V}{\partial\theta} + \frac{n_\nu}{n_\zeta}\frac{\partial V}{\partial\nu}\Big).$$

Using the preceding estimates uniform with respect to ν and θ, we obtain the existence of $\kappa > 0$ independant of ε and ν such that for ε small enough and for any $\nu \in I$,

$$\Big|\int_0^{2\pi}\gamma^\varepsilon(\nu,\theta)\frac{\partial V}{\partial n}(\nu,\theta)\,d\theta - \int_0^{2\pi}\widehat{\gamma}_\nu(\theta)\frac{\partial\widehat{V}_\nu}{\partial\widehat{n}_\nu}(\theta)\,d\theta\Big| < \kappa\,\varepsilon.$$

According to lemma 4.2, we know that $\int_0^{2\pi}\widehat{\gamma}_\nu(\theta)\frac{\partial\widehat{V}_\nu}{\partial\widehat{n}_\nu}(\theta)\,d\theta = 2\pi$, hence the desired result.

■

4.3 Expansion in the neighborhood of infinity

4.3.1 Formal expansion

We are now going to look for an expansion of $U^\varepsilon_{1,\nu}$ in the neighborhood of infinity. It suffices to obtain such an expansion for V_ν. Since Φ is a C^∞ function, it is tedious but easy computation to show that $\nu \mapsto \widehat{V}_\nu$ is C^∞ on $[-1;+1]$ as a function valuated in $W^{1,-1}(\widehat{\mathcal{E}}_\perp \setminus \overline{D_{R_N}})$. In $\widehat{\mathcal{E}}_\perp \setminus \overline{D_{R_N}}$, $\widehat{V}_\nu$ can be decomposed using separation of variables in the following manner

$$\widehat{V}_\nu(\zeta) = \ln\zeta + \sum_{p=-\infty}^{+\infty}\frac{V_{\nu,p}}{\zeta^{|p|}}\;e^{ip\theta}$$

Moreover for each $p \in \mathbb{Z}$, it is easy to show the continuous dependancy of $V_{\nu,p}$ with respect to ν

$$\int_{R_N}^{2R_N}\int_0^{2\pi}\widehat{V}_\nu(\zeta;\theta)\frac{e^{-ip\theta}}{\zeta}\,d\zeta d\theta = \frac{1}{|p|}\frac{1}{R_N^{|p|}}(1 - \frac{1}{2^{|p|}})\,2\pi\,V_{\nu,p}$$

From this we deduce the existence of a constant $\kappa > 0$ independant of ν such that

$$\forall\nu \in [-1;+1],\ \forall p \in \mathbb{Z},\ |V_{\nu,p}| \leqslant \kappa\,R_N^{|p|} \tag{10}$$

Derivating with respect to ν in the identity (10), we obtain the existence of another constant $\kappa > 0$ independant of ν such that

$$\forall\nu \in [-1;+1],\ \forall p \in \mathbb{Z}, \quad |\frac{d}{d\nu}V_{\nu,p}| \leqslant \kappa\,R_N^{|p|} \quad \text{and} \quad |\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}V_{\nu,p}| \leqslant \kappa\,R_N^{|p|} \tag{11}$$

Let us finish this section by writing the first two terms of the expansion in the neighborhood of infinity of the function U^ε_1 we have just constructed. If we denote $b^\varepsilon(\nu) = b^\varepsilon_\nu$ and $V_0 :\ \nu \mapsto V_{\nu,0}$, then according to the preceding results $U^\varepsilon_1(\mathbf{x}) = b^\varepsilon(\nu)\ln\zeta + b^\varepsilon(\nu)V_0(\nu) + o(1)$. As a consequence

$$U^\varepsilon_{1,as}(\mathbf{x}) = b^\varepsilon(\nu)\ln\zeta + b^\varepsilon(\nu)V_0(\nu).$$

4.3.2 Error estimates

Like in the section dedicated to the far field, we prove estimates for the near field. As a consequence we will propose estimates for the following quantities

$$\|U_1^\varepsilon - U_{1,as}^\varepsilon\|_{L_\xi^\infty \otimes L_\nu^2(\mathcal{T}^\varepsilon)} \quad \|\frac{\partial}{\partial \xi}\{U_1^\varepsilon - U_{1,as}^\varepsilon\}\|_{L_\xi^\infty \otimes L_\nu^2(\mathcal{T}^\varepsilon)} \quad \text{and} \quad \|\Delta U_1^\varepsilon\|_{L^2(\mathcal{Z}_{\mathrm{pr}}^\varepsilon)}$$

- **estimate of** $\|U_1^\varepsilon - U_{1,as}^\varepsilon\|^2_{L_\xi^\infty \otimes L_\nu^2(\mathcal{T}^\varepsilon)}$
  We obtain directly
$$U_1^\varepsilon - U_{1,as}^\varepsilon = b^\varepsilon(\nu) \sum_{|p|\geqslant 1} \frac{V_{\nu,p}}{\zeta^{|p|}}\, e^{ip\theta}$$
  According to the inequality (10) and for ε small enough, we can find $\kappa > 0$ independant of ν and p such that, for $\zeta > R_N$
$$\Big| \sum_{|p|\geqslant 1} \frac{V_{\nu,p}}{\zeta^{|p|}}\, e^{ip\theta} \Big| \leqslant \frac{R_N}{\zeta} \sum_{|p|\geqslant 1} \frac{|V_{\nu,p}|}{R_N\, \zeta^{|p|-1}} \leqslant \frac{\kappa}{\zeta} \leqslant \kappa\, \varepsilon^{1/2}.$$
  Hence the estimate
$$\|U_1^\varepsilon - U_{1,as}^\varepsilon\|^2_{L_\xi^\infty \otimes L_\nu^2(\mathcal{T}^\varepsilon)} \leqslant \kappa\ \|b^\varepsilon\|_{E^0(I)}\ \varepsilon^{1/2}$$

- **estimate of** $\|\frac{\partial}{\partial \xi}\{U_1^\varepsilon - U_{1,as}^\varepsilon\}\|^2_{L_\xi^\infty \otimes L_\nu^2(\mathcal{T}^\varepsilon)}$
  We have
$$\frac{\partial}{\partial \xi}\{U_1^\varepsilon - U_{1,as}^\varepsilon\} = \frac{\xi}{\varepsilon^2 \zeta}\, b^\varepsilon(\nu) \sum_{|p|\geqslant 1} -|p|\, \frac{V_{\nu,p}}{\zeta^{|p|+1}}\, e^{ip\theta}$$
  Again using inequality (10) we obtain $\kappa > 0$ independant of ν such that for ε small enough
$$\Big| \sum_{|p|\geqslant 1} -|p|\, \frac{V_{\nu,p}}{\zeta^{|p|+1}}\, e^{ip\theta} \Big| \leqslant \frac{R_N}{\zeta^2} \sum_{|p|\geqslant 1} |p|\, \frac{|V_{\nu,p}|}{R_N\, \zeta^{|p|-1}} \leqslant \frac{\kappa}{\zeta^2} \leqslant \kappa\, \varepsilon.$$
  Taking into account the fact that $|\frac{\xi}{\varepsilon^2\zeta}| < \frac{2}{\varepsilon^{3/2}}$, we are led to
$$\|\frac{\partial}{\partial \xi}\{U_1^\varepsilon - U_{1,as}^\varepsilon\}\|_{L_\xi^\infty \otimes L_\nu^2(\mathcal{T}^\varepsilon)} \leqslant \kappa\, \frac{1}{\varepsilon^{1/2}}$$

- **estimate of** $\|\Delta U_1^\varepsilon\|_{L^2(\mathcal{Z}_{\mathrm{pr}}^\varepsilon)}$
  Using (10) and (11), some computations yield the existence of $\kappa > 0$ independant of ε, ν and θ such that
$$\begin{aligned}
&|\zeta \frac{\partial}{\partial \zeta} \zeta \frac{\partial U_1^\varepsilon}{\partial \zeta}|^2 \leqslant \kappa^2\, |b^\varepsilon(\nu)|^2 \\
&|\zeta \frac{\partial U_1^\varepsilon}{\partial \zeta}|^2 \leqslant \kappa^2\, |b^\varepsilon(\nu)|^2 \ln^2 \frac{1}{\varepsilon} \\
&|\frac{\partial}{\partial \nu}(1-\nu^2)\frac{\partial U_1^\varepsilon}{\partial \nu}|^2 \leqslant \kappa^2 \left(|b^\varepsilon(\nu)|^2 + (1-\nu^2)|\frac{d\, b^\varepsilon}{d\nu}|^2 + |\frac{d}{d\nu}(1-\nu^2)\frac{d\, b^\varepsilon}{d\nu}|^2 \right) \ln^2 \frac{1}{\varepsilon} \\
&|k^2(1-\nu^2) U_1^\varepsilon|^2 \leqslant \kappa^2\, |b^\varepsilon(\nu)|^2\, \ln^2 \frac{1}{\varepsilon} \\
&|k^2 \varepsilon^2 \zeta^2 U_1^\varepsilon|^2 \leqslant \kappa^2\, |b^\varepsilon(\nu)|^2
\end{aligned}$$

Moreover, according to the assumption that $\Phi(\nu;\theta) = \Phi(\nu)$ when $|\nu| \geqslant \nu_0$, we obtain that for $|\nu| \geqslant \nu_0$, U_1^ε is independant of θ. Again using (10) and (11), we obtain the existence of $\kappa > 0$ independant of ε, ν and θ such that

$$|\frac{1}{1-\nu^2}\frac{\partial^2 U_1^\varepsilon}{\partial\theta^2}|^2 \leqslant \kappa^2\,|b^\varepsilon(\nu)|^2$$

From the preceding inequalities, we can deduce the following ones

$$\|\Delta U_1^\varepsilon\|^2_{L^2(\mathcal{Z}^\varepsilon_{\mathrm{pr}})} \leqslant \kappa^2\,\|b^\varepsilon\|_{E^2(I)}\ln^2\frac{1}{\varepsilon}\int_{\nu=-1}^{+1}\int_{\theta=0}^{2\pi}\int_{\zeta=\Phi(\nu;\theta)}^{\sqrt{2}/\varepsilon^{1/2}}\varepsilon^2\frac{\zeta}{\xi}d\zeta d\nu d\theta \leqslant \kappa^2\,\|b^\varepsilon\|^2_{E^4(I)}\,\varepsilon\ln^2\frac{1}{\varepsilon}.$$

To sum up we have the estimate

$$\|\Delta U_1^\varepsilon\|_{L^2(\mathcal{Z}^\varepsilon_{\mathrm{pr}})} \leqslant \kappa\,\|b^\varepsilon\|^2_{E^2(I)}\,\varepsilon^{1/2}\,\ln\frac{1}{\varepsilon}$$

5 Step 3: matching of near and far fields

With (1), (2), (8) et (9) we defined two functions that must represent u^ε in two zones: (2) in the near field zone and (1) in the far field zone. Let us formulate a remark concerning the functions (2) and (1). We did not fully define them, and left and indeterminacy on a^ε and b^ε. We will now show that the pair $(a^\varepsilon; b^\varepsilon)$ has to be a solution of an integral equation in order to guaranty that (2) and (1) suitably describe the same field u^ε respectively in $\mathcal{Z}^\varepsilon_{\mathrm{pr}}$ and $\mathcal{Z}^\varepsilon_{\mathrm{lo}}$. Indeed if this is the case, then $u_0 + u_1^\varepsilon$ has to be close to U_1^ε in $\mathcal{T}^\varepsilon$. This is what we write now in order to obtain the equation satisfied by a^ε and b^ε,

$$u_0(\mathbf{x}) + u_1^\varepsilon(\mathbf{x}) - U_1^\varepsilon(\mathbf{x}) \underset{\substack{\varepsilon\to 0\\ \mathbf{x}\in\mathcal{T}^\varepsilon}}{\longrightarrow} 0$$

Now replacing the terms in the preceding identity by their respective expansion, and replacing "$\to 0$" by "$= 0$", we obtain:

$$u_{0,as}(\mathbf{x}) + u^\varepsilon_{1,as}(\mathbf{x}) - U^\varepsilon_{1,as}(\mathbf{x}) = 0$$

$$u_{0,as}(\nu) + \frac{a^\varepsilon(\nu)}{4\pi}\ln\Big(\frac{2}{\xi-1}\Big) + \frac{1}{4\pi}\int_{-1}^{+1}\frac{a^\varepsilon(z)-a^\varepsilon(\nu)}{|z-\nu|}dz + \frac{1}{4\pi}\int_{-1}^{+1}a^\varepsilon(z)\frac{e^{ik|\nu-z|}-1}{|\nu-z|}dz$$
$$+\int_{-1}^{+1}a^\varepsilon(z)G^{\mathrm{reg}}(\nu;z)dz - b^\varepsilon(\nu)\ln\frac{\sqrt{2(\xi-1)}}{\varepsilon} - b^\varepsilon(\nu)V_0(\nu) = 0$$

This equation is composed of terms constant or logarithmically varying with ξ. As a consequence the following two equalities have to be satisfied

$$\frac{a^\varepsilon(\nu)}{4\pi} + \frac{b^\varepsilon(\nu)}{2} = 0$$

$$u_{0,as}(\nu) + \frac{a^\varepsilon(\nu)}{4\pi}\ln(2) + \frac{1}{4\pi}\int_{-1}^{+1}\frac{a^\varepsilon(z)-a^\varepsilon(\nu)}{|z-\nu|}dz + \frac{1}{4\pi}\int_{-1}^{+1}a^\varepsilon(z)\frac{e^{ik|\nu-z|}-1}{|\nu-z|}dz$$
$$+\int_{-1}^{+1}a^\varepsilon(z)G^{\mathrm{reg}}(\nu;z)dz - \frac{b^\varepsilon(\nu)}{2}\ln\Big(\frac{2}{\varepsilon^2}\Big) - b^\varepsilon(\nu)V_0(\nu) = 0$$

These equations are matching conditions of the expansion. We can simplify them using the first equation and putting it in the second one. We are led to a single integral equation. Define $\frac{1}{2}\ln c(\nu)^2 = \ln(4) + V_0(\nu)$ and $v_0(\nu) = -4\pi\,u_{0,as}(\nu)$. We obtain the equation $\forall\nu\in[-1;+1]$

$$\begin{aligned}\ln(\frac{c(\nu)^2}{\varepsilon^2})\,a^\varepsilon(\nu) + \int_{-1}^{+1}\frac{a^\varepsilon(z)-a^\varepsilon(\nu)}{|z-\nu|}dz\\ +\int_{-1}^{+1}a^\varepsilon(z)\frac{e^{ik|\nu-z|}-1}{|\nu-z|}dz + \int_{-1}^{+1}4\pi\,a^\varepsilon(z)G^{\mathrm{reg}}(\nu;z)dz = v_0\end{aligned} \tag{12}$$

We will study this equation in detail. We rewrite equation (12) in the following manner

$$\Big(\ln\frac{1}{\varepsilon^2}.Id + A + B\Big)a^\varepsilon = v_0$$

A, B et Id are linear operator (A is unbounded) of $L^2(I) \to L^2(I)$ defined by

$$\begin{cases} \Big(Au\Big)(\nu) = \displaystyle\int_{-1}^{+1} \frac{u(z)-u(\nu)}{|z-\nu|}dz \\ B = B_1 + B_2 + B_3 \quad \text{avec} \\ \Big(B_1 u\Big)(\nu) = u(\nu)\ln c(\nu)^2 \\ \Big(B_2 u\Big)(\nu) = \displaystyle\int_{-1}^{+1} u(z)\frac{e^{ik|\nu-z|}-1}{|\nu-z|}dz \\ \Big(B_3 u\Big)(\nu) = \displaystyle\int_{-1}^{+1} 4\pi\, u(z)\, G^{\text{reg}}(\nu;z)\,dz \end{cases}$$

Actually we prove in the appendix of this document that B_1, B_2 and B_3 are bounded ($L^2(I)$ supplied with the usual scalar product), whereas A appears to be unbounded. As a consequence we will adapt our functionl setting to A.

5.1 Study of A

In order to study the operator A, one relevant manner consists in finding a diagonalizing basis. We are lucky! Such a basis exists and is explicit: the family of Legendre polynomials (a detailed introduction to Legendre polynomials is available in the appendix A.2). This implies in particular that Legendre polynomials will play a central role in this study. This is due to the fact that the principal part of the Helmholtz operator i.e. the Laplace operator is solvable by separation of variable in ellipsoidal coordinates using Legendre polynomials.

Lemma 5.1
If P_n is the order n Legendre polynomial and W_{n-1} is defined by the formula (20) given in appendix A.2, then

$$(A\,P_n)(\nu) = 2\,W_{n-1}(1)\,P_n(\nu).$$

Proof:
We use lemma A.1 established in the appendix A.2 dedicated to Legendre polynomials.

$$\begin{aligned}\int_{-1}^{+1}\frac{P_n(z)\,dz}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}} &= \int_{-1}^{+1}\frac{P_n(z)-P_n(\nu)}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}}\,dz + P_n(\nu)\ln\Big(\frac{\xi+1}{\xi-1}\Big)\\ &= 2P_n(\nu)\,Q_n(\xi)\\ &= P_n(\nu)\ln\Big(\frac{\xi+1}{\xi-1}\Big) + 2W_{n-1}(1)P_n(\nu)\\ &\quad + P_n(\nu)\Big(P_n(\xi)-P_n(1)\Big)\ln\Big(\frac{\xi+1}{\xi-1}\Big)\\ &\quad + 2\,P_n(\nu)\Big(W_{n-1}(\xi)-W_{n-1}(1)\Big)\end{aligned}$$

Besides it is clear that

$$\Big(P_n(\xi)-P_n(1)\Big)\ln\Big(\frac{\xi+1}{\xi-1}\Big)\underset{\xi\to 1_+}{\to} 0 \quad\text{and}\quad W_{n-1}(\xi)-W_{n-1}(1)\underset{\xi\to 1_+}{\to} 0.$$

In addition, note that $(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2 \geqslant (\nu-z)^2$, so domainated convergence yields

$$\int_{-1}^{+1} \frac{P_n(z)-P_n(\nu)}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}}\,dz \underset{\xi\to 1_+}{\to} (A\,P_n)(\nu).$$

This gives the desired result.

■

Thus for $n\in\mathbb{N}$, P_n is an eigenvector A associated with the eigenvalue $2W_{n-1}(1)$. Let $\widetilde{P}_n = CP_n$ with C chosen such that $\|\widetilde{P}_n\|_{L^2(I)} = 1$. Since $(\widetilde{P}_n)_{n\in\mathbb{N}}$ is an orthonormal basis whose linear combination are dense in $L^2(I)$, we have obtained a diagonalizing basis for A. The spectrum of A denoted $\sigma(A)$ is given by

$$\sigma(A) = \{2W_{n-1}(1)\ ,\ n\in\mathbb{N}\} \quad \text{with} \quad W_{n-1}(1) = \sum_{k=1}^{n} \frac{-1}{k} \quad \text{defining} \quad W_{-1}(1) = 0.$$

Note that $(W_{n-1}(1))_{n\in\mathbb{N}}$ is very well known. In particular, we have the classical result

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1) \qquad \text{when} \quad n\to+\infty$$

Define $\lambda_n = \exp(W_{n-1}(1))$. Simple computations yields

$$2W_{n-1}(1) = \ln\lambda_n^2\ ,\ \ \lambda_n^2 \underset{n\to+\infty}{\sim} \frac{e^{-2\gamma}}{n^2}\ ,\ \ \lambda_n^2 \geqslant \frac{e^{-2}}{(n+1)^2} \tag{13}$$

Then we obtain

$$\begin{aligned} D(A) &= \{u\in L^2(I)\ |\ \sum_{n=0}^{+\infty} |\ln\lambda_n^2|^2\, |\left\langle u;\widetilde{P}_n\right\rangle|^2\ < +\infty\} \\ Au &= \sum_{n=0}^{+\infty} \ln\lambda_n^2\,\left\langle u;\widetilde{P}_n\right\rangle_{L^2(I)} \widetilde{P}_n \qquad ,\quad \forall u\in D(A) \end{aligned}$$

The convergence of the sum above holds in $L^2(I)$.

5.2 Functional setting adapted to A

Since we know in detail the operator A, we introduce a suitable functional setting. We will use the Sobolev spaces on the unit sphere. Legendre polynomials naturally appear in that setting, and in the study of the Legendre operator which is another form of the Laplace Beltrami on the sphere.

$$\begin{aligned} \mathcal{L}:\ \ D(\mathcal{L}) \ &\to\ L^2(I) \\ u\ &\mapsto\ \frac{d}{d\nu}(1-\nu^2)\frac{du}{d\nu} \end{aligned}$$

Indeed we recall in appendix A.2, that Legendre polynomials diagonalize the operator $\mathcal{L}$, $\partial_\nu(1-\nu^2)\partial_\nu P_l = -l(l+1)P_l$. Recalling that Sobolev spaces of integral order on the interval I can be defined by $H^n(I) = \{u\in L^2(I)\ |\ \left\langle (-\frac{d^2}{d\nu^2})^k u; u\right\rangle_{L^2(I)} < +\infty \quad k = 0\ldots n\}$. We introduce in the same manner the spaces

$$\begin{aligned} E^n(I) \ &= \{u\in L^2(I)\ |\ \langle(-\mathcal{L})^k u;u\rangle_{L^2(I)} < +\infty \quad k=0\ldots n\} \\ &= \{u\in L^2(I)\ |\ \sum_{l=0}^{+\infty}(1+l^2)^n |\left\langle u;\widetilde{P}_l\right\rangle_{L^2(I)}|^2 < +\infty \quad \}. \end{aligned}$$

We can generalize this definition like in the case of Sobolev spaces $\forall s \in \mathbb{R}$, $E^s(I) = \{u \in L^2(I) \mid \sum_{l=0}^{+\infty}(1+l^2)^s|\langle u; P_l\rangle_{L^2(I)}|^2 < +\infty \}$. We define a natural norm for the space $E^s(I)$,

$$\forall u \in E^s(I), \quad \|u\|^2_{E^s(I)} = \sum_{l=0}^{+\infty}(1+l^2)^s|\left\langle u; \widetilde{P}_l\right\rangle_{L^2(I)}|^2.$$

Using the expression of the Laplace operator on the sphere given in paragraph 2.2.1, we see that a function $u \in E^s(I)$ van be identify with a function $\widetilde{u} \in H^s(S^2)$ such that $\partial_\theta \widetilde{u} = 0$. According to this remark we can formulate the lemma,

Lemma 5.2

$$\textit{For } n \geqslant 1\ ,\ E^n(I) = H^n(S^2) \cap Ker\frac{\partial}{\partial\theta}.$$

Note that the statement of this lemma contains abusive notation since it contains an implicit change of variables that transforms I into S^2: we identified the coordinate ν repairing a point of I with the coordinate ν on S^2.

This result clearly suggest to look at J as a sphere (!). This idea already appeared in lemma 2.1 which led us to caracterize $H^{1/2}(\Gamma^\varepsilon)$ by mapping Γ^ε on the sphere. Here we are not interested in Γ^ε (2 dimensional sub-manifold) but in J (one dimensional-submanifold). However we suggest to keep in mind that $\lim_{\varepsilon\to 0}\Gamma^\varepsilon = J$.

5.3 Existence of an approximate solution for the matching equation

We now look for a solution to the equation (12). The problem is that the operator associated with this equation is not positive (in $L^2(I)$). Indeed in the case where ε is such that there exists n with $\varepsilon = \lambda_n$, then $\mathrm{Ker}(\ln\frac{1}{\varepsilon^2}\,Id + A) \neq \{0\}$. The matching equation is rather of the form coercive + compact so this problem shall admit resonnace frequencies depending on ε. This is not tolerable because we need to define a function a^ε for any value of ε. Thus we will use a regularization trick. Define

$$\begin{aligned}
\mathfrak{E}^\varepsilon &= \{u \in L^2(I) \mid \langle u; P_q\rangle_{L^2(I)} = 0 \text{ for } q > [1/\sqrt{\varepsilon}]\ \} \\
\pi^\varepsilon :\quad L^2(I) &\to \mathfrak{E}^\varepsilon \\
u &\mapsto \sum_{q=0}^{[1/\sqrt{\varepsilon}]}\left\langle u; \widetilde{P}_q\right\rangle_{L^2(I)}\widetilde{P}_q
\end{aligned}$$

For an $n \in \mathbb{N}$, $\mathfrak{E}^\varepsilon$ is a closed sub-space of $E^n(I)$ (supplied with the norm $\|\ \|_{E^n(I)}$) so $\pi^\varepsilon_{|E^n(I)}$ is continuous as an operator from $E^n(I)$ into $E^n(I)$. For any $n \in \mathbb{N}$, let us denote $\|\ \|_n$ the norm induced by $E^n(I)$ on $\mathfrak{E}^\varepsilon$. Thus if $T : \mathfrak{E}^\varepsilon \to \mathfrak{E}^\varepsilon$ is alinear operator, then

$$\|T\|_n \ = \ \sup\left\{\|Tx\|_{E^n(I)} \mid x \in \mathfrak{E}^\varepsilon \text{ et } \|x\|_{E^n(I)} = 1\right\}$$

Note that $\forall\varepsilon$, A maps $\mathfrak{E}^\varepsilon$ into itself. Now define

$$\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon \ = \ \left(\ln\frac{1}{\varepsilon^2}\,Id + A\right)_{|\mathfrak{E}^\varepsilon}$$

Instead of solving (12), we will rather solve

$$(12)_\varepsilon : \begin{cases} \text{Find } a^\varepsilon \in \mathfrak{E}^\varepsilon \text{ such that} \\[2ex] \left(\ln\dfrac{1}{\varepsilon^2}\,Id + A^\varepsilon\right)a^\varepsilon + \pi^\varepsilon B a^\varepsilon \ = \ \pi^\varepsilon v_0 \end{cases}$$

Theorem 5.1
For $\varepsilon > 0$ small enough, there exists a unique $a^\varepsilon \in \mathfrak{E}^\varepsilon$ such that

$$\Big(\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon\Big)a^\varepsilon + \pi^\varepsilon B a^\varepsilon \;=\; \pi^\varepsilon v_0.$$

In addition a^ε satisfies

$$\|a^\varepsilon\|_{E^n(I)} \leqslant \frac{2\|v_0\|_{E^n(I)}}{\ln(\dfrac{e^{-2\gamma}}{\varepsilon})}.$$

Proof:
We have the explicit formula: $\forall u \in \mathfrak{E}^\varepsilon$

$$\Big(\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon\Big)u \;=\; \sum_{q=0}^{[1/\sqrt{\varepsilon}]} \ln\Big(\frac{\lambda_q}{\varepsilon}\Big)^2 \Big\langle u;\widetilde{P}_q\Big\rangle_{L^2(I)}\widetilde{P}_q.$$

We can consider its inverse as an operator mapping $\mathfrak{E}^\varepsilon$ to $\mathfrak{E}^\varepsilon$:

$$\Big(\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon\Big)^{-1}u \;=\; \sum_{q=0}^{[1/\sqrt{\varepsilon}]} \frac{1}{\ln\Big(\dfrac{\lambda_q}{\varepsilon}\Big)^2} \Big\langle u;\widetilde{P}_q\Big\rangle_{L^2(I)}\widetilde{P}_q.$$

Using the computational remarks (13), we obtain for ε small enough,

$$\frac{\lambda_q}{\varepsilon} = \frac{1}{e\,(q+1)\,\varepsilon} \geqslant \frac{1}{e\,([1/\sqrt{\varepsilon}]+1)\,\varepsilon} \geqslant \frac{1}{2e\,\sqrt{\varepsilon}} \quad\Rightarrow\quad \frac{1}{\ln(\frac{\lambda_q}{\varepsilon})^2} \leqslant \frac{1}{\ln(\frac{e^{-2}}{4\,\varepsilon})}$$

and this provides an estimate for the spectral radius

$$\rho\Big((\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon)^{-1}\Big) \leqslant \frac{1}{\ln(\dfrac{e^{-2}}{4\varepsilon})}.$$

Since for all the scalar products induced by $\langle\;;\;\rangle_{E^n(I)}$ on $\mathfrak{E}^\varepsilon$, $(\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon)^{-1}$ is normal. As a consequence, $\forall n \in \mathbb{N}$:

$$\|(\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon)^{-1}\|_n = \rho\Big((\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon)^{-1}\Big) \leqslant \frac{1}{\ln(\dfrac{e^{-2}}{4\varepsilon})}.$$

The equation $(12)_\varepsilon$ can be multiplied on the left by $(\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon)^{-1}$ and we are thus led to

$$\Big(Id + (\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon)^{-1}\pi^\varepsilon B\Big)a^\varepsilon \;=\; (\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon)^{-1}\pi^\varepsilon v_0$$

with

$$\|(\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon)^{-1}\pi^\varepsilon B\|_n \;\leqslant\; \frac{\|B\|_{E^n(I)}}{\ln(\dfrac{e^{-2}}{4\varepsilon})}$$

where

$$\|B\|_{E^n(I)} \;=\; \sup\Big\{\|Bx\|_{E^n(I)} \mid x \in E^n(I) \text{ and } \|x\|_{E^n(I)} = 1\Big\}$$

is independant of ε. So for ε small enough $Id + (\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon)^{-1}\pi^\varepsilon B$ is invertible and for n fixed and ε small enough

$$\|(Id + (\ln\frac{1}{\varepsilon^2}\,Id + A^\varepsilon)^{-1}\pi^\varepsilon B)^{-1}\|_n \leqslant 2.$$

Since in finite dimension, invertibility is independant of the topology,

$$a^\varepsilon \;=\; \Big(Id + (\ln \frac{1}{\varepsilon^2}\, Id + A^\varepsilon)^{-1}\pi^\varepsilon B\Big)^{-1} (\ln \frac{1}{\varepsilon^2}\, Id + A^\varepsilon)^{-1}\pi^\varepsilon v_0$$

is independant of the norm chosen for $\mathfrak{E}^\varepsilon$. Thus $\forall n \in \mathbb{N}$, $a^\varepsilon \in \mathfrak{E}^\varepsilon(I)$ and

$$\|a^\varepsilon\|_{E^n(I)} \leqslant \frac{2\|v_0\|_{E^n(I)}}{\ln(\dfrac{e^{-2}}{4\varepsilon})}.$$

■

Now we show that a^ε constructed with lemma 5.1 satisfies approximately equation (12).

Lemma 5.3
There exists $\tau^\varepsilon \in \cap_{n\in\mathbb{N}} E^n(I)$ *such that*

$$(\ln \frac{1}{\varepsilon^2} Id + A + B)a^\varepsilon = v_0 + \tau^\varepsilon$$

and $\forall r, s \in \mathbb{N}$, $\exists\, \kappa > 0$ *such that* $\|\tau^\varepsilon\|_{E^r(I)} \leqslant \kappa\, \varepsilon^s$.

Proof:

$$\begin{aligned}
(\ln \frac{1}{\varepsilon^2} Id + A + B)a^\varepsilon &= (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon + \pi^\varepsilon B)a^\varepsilon + (Id - \pi^\varepsilon)a^\varepsilon \\
&= \pi^\varepsilon v_0 + (Id - \pi^\varepsilon)Ba^\varepsilon \\
&= v_0 + \underbrace{(Id - \pi^\varepsilon)Ba^\varepsilon - (Id - \pi^\varepsilon)v_0}_{=\,\tau^\varepsilon}.
\end{aligned}$$

It is clear, according to the smoothness of u_0, that $v_0 \in \cap_{n\in\mathbb{N}} E^n(I)$ and since $a^\varepsilon \in \cap_{n\in\mathbb{N}} E^n(I)$ and B maps $E^n(I)$ into itself for any $n \in \mathbb{N}$, $Ba^\varepsilon - v_0 \in \cap_{n\in\mathbb{N}} E^n(I)$. Moreover sinv=ce $\mathfrak{E}^\varepsilon \subset \cap_{n\in\mathbb{N}} E^n(I)$, it is clear that $(Id - \pi^\varepsilon)E^n(I) \subset E^n(I)$ for any n.

To sum up: $\tau^\varepsilon \in \cap_{n\in\mathbb{N}} E^n(I)$

Now we choose r and $s \in \mathbb{N}$ and show separately that (which yields the conclusion):

$$(i)\ \exists\, \kappa_1 > 0 \quad \text{s.t} \quad \|(Id - \pi^\varepsilon)u_0\|_{E^r(I)} \leqslant \kappa_1 \varepsilon^s,$$

$$(ii)\ \exists\, \kappa_2 > 0 \quad \text{s.t} \quad \|(Id - \pi^\varepsilon)Ba^\varepsilon\|_{E^r(I)} \leqslant \kappa_2 \varepsilon^s.$$

First of all,

$$\begin{aligned}
\|(Id - \pi^\varepsilon)u_0\|^2_{E^r(I)} &= \sum_{q=0}^{r} \langle(-\mathcal{L})^q (Id - \pi^\varepsilon)u_0; (Id - \pi^\varepsilon)u_0\rangle_{L^2(I)} \\
&= \sum_{q=0}^{r} \sum_{l=[1/\sqrt{\varepsilon}]+1}^{+\infty} \Big\langle (-\mathcal{L})^q u_0; \widetilde{P}_l \Big\rangle_{L^2(I)} \overline{\Big\langle u_0; \widetilde{P}_l \Big\rangle}_{L^2(I)} \\
&= \sum_{q=0}^{r} \sum_{l=[1/\sqrt{\varepsilon}]+1}^{+\infty} \frac{1}{l^{4s}(l+1)^{4s}} \Big\langle (-\mathcal{L})^{q+2s} u_0; \widetilde{P}_l \Big\rangle_{L^2(I)} \overline{\Big\langle (-\mathcal{L})^{2s} u_0; \widetilde{P}_l \Big\rangle}_{L^2(I)} \\
&\leqslant \Big\{ \sum_{l=[1/\sqrt{\varepsilon}]+1}^{+\infty} \frac{1}{l^{4s}(l+1)^{4s}} \Big\} \Big\{ \sum_{q=0}^{r} \|(-\mathcal{L})^{q+2s} u_0\|_{L^2(I)} \|(-\mathcal{L})^{2s} u_0\|_{L^2(I)} \Big\} \\
&\leqslant \frac{\varepsilon^{3s}}{6s} \sum_{q=0}^{r} \|(-\mathcal{L})^{q+2s} u_0\|_{L^2(I)} \|(-\mathcal{L})^{2s} u_0\|_{L^2(I)}.
\end{aligned}$$

which yields (i). With similar computation we obtain

$$\begin{aligned}
\|(Id-\pi^{\varepsilon})Ba^{\varepsilon}\|^2_{E^r(I)} & \leqslant \frac{\varepsilon^{3s}}{6s}\sum_{q=0}^{r}\|(-\mathcal{L})^{q+2s}Ba^{\varepsilon}\|_{L^2(I)}\|(-\mathcal{L})^{2s}Ba^{\varepsilon}\|_{L^2(I)}\\
& \leqslant \frac{\varepsilon^{3s}}{6s}\sum_{q=0}^{r}\|B\|_{E^{2q+4s}(I)}\,\|a^{\varepsilon}\|_{E^{2q+4s}(I)}\|B\|_{E^{4s}(I)}\|a^{\varepsilon}\|_{E^{4s}(I)}\\
& \leqslant \Big(\frac{2}{\ln(\frac{e^{-2\gamma}}{\varepsilon})}\Big)^2\,\frac{\varepsilon^{3s}}{6s}\sum_{q=0}^{r}\|B\|_{E^{2q+4s}(I)}\,\|v_0\|_{E^{2q+4s}(I)}\|B\|_{E^{4s}(I)}\|v_0\|_{E^{4s}(I)}.
\end{aligned}$$

Which concludes the proof by showing (ii).

■

5.4 Error estimate for the matching

Like in the preceding two sections, we will now provide estimates related to the matching condition. We formulate and prove a result that allows to know precisely the behavior of a^{ε} when $\varepsilon\to 0$. In a first step we will estimate $\|u_0+u_1^{\varepsilon}-U_1^{\varepsilon}\|^2_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}$ and $\|\frac{\partial}{\partial\xi}\{u_0+u_1^{\varepsilon}-U_1^{\varepsilon}\}\|^2_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}$. According to the calculus provided before,

$$u_{0,as}+u^{\varepsilon}_{1,as}-U^{\varepsilon}_{1,as}=\Big(\ln\frac{1}{\varepsilon^2}\,Id+A+B\Big)a^{\varepsilon}-b^{\varepsilon}(\nu)\ln\sqrt{\frac{\xi+1}{2}}=\tau^{\varepsilon}-\frac{b^{\varepsilon}(\nu)}{2}\ln(1+\frac{\xi-1}{2}).$$

Using the results on τ^{ε}, we obtain the existence of $\kappa>0$ such that

$$\|u_{0,as}+u^{\varepsilon}_{1,as}-U^{\varepsilon}_{1,as}\|^2_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}\leqslant\kappa\,\|a^{\varepsilon}\|_{E^3(I)}\,\varepsilon.$$

In conclusion we have

$$\begin{aligned}
\|u_0+u_1^{\varepsilon}-U_1^{\varepsilon}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})} & \leqslant \|u_0-u_{0,as}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}+\|u_1^{\varepsilon}-u^{\varepsilon}_{1,as}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}\\
& \quad +\|U_1^{\varepsilon}-U^{\varepsilon}_{1,as}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}+\|u_{0,as}+u^{\varepsilon}_{1,as}-U^{\varepsilon}_{1,as}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}.\\
\|u_0+u_1^{\varepsilon}-U_1^{\varepsilon}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})} & \leqslant \kappa\,\|a^{\varepsilon}\|_{E^3(I)}\,\varepsilon^{1/2}.
\end{aligned}$$

Then differentiating the expression above

$$\frac{\partial}{\partial\xi}\{u_{0,as}+u^{\varepsilon}_{1,as}-U^{\varepsilon}_{1,as}\}=-\frac{b^{\varepsilon}(\nu)}{2}\frac{1}{\xi+1}$$

hence the existence of a $\kappa>0$ independant of ε such that

$$\|\frac{\partial}{\partial\xi}\{u_{0,as}+u^{\varepsilon}_{1,as}-U^{\varepsilon}_{1,as}\}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}\leqslant\kappa\,\|a^{\varepsilon}\|_{E^3(I)}.$$

Finally

$$\begin{aligned}
\|\frac{\partial}{\partial\xi}\{u_0+u_1^{\varepsilon}-U_1^{\varepsilon}\}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})} & \leqslant \|\frac{\partial}{\partial\xi}\{u_0-u_{0,as}\}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}+\|\frac{\partial}{\partial\xi}\{u_1^{\varepsilon}-u^{\varepsilon}_{1,as}\}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}\\
& \quad +\|\frac{\partial}{\partial\xi}\{U_1^{\varepsilon}-U^{\varepsilon}_{1,as}\}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}\\
& \quad +\|\frac{\partial}{\partial\xi}\{u_{0,as}+u^{\varepsilon}_{1,as}-U^{\varepsilon}_{1,as}\}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})}.\\
\|\frac{\partial}{\partial\xi}\{u_0+u_1^{\varepsilon}-U_1^{\varepsilon}\}\|_{L^{\infty}_{\xi}\otimes L^2_{\nu}(\mathcal{T}^{\varepsilon})} & \leqslant \kappa\,\|a^{\varepsilon}\|_{E^3(I)}\,\frac{1}{\varepsilon^{1/2}}.
\end{aligned}$$

Now we formulate a result that gives the behavior of a^{ε}. This estimate will lead us to the behavior of $\widetilde{p}^{\varepsilon}$ in subsection 8.2.

Lemma 5.4
There exists two constants $\kappa_1, \kappa_2 > 0$ independant of ε, such that for ε small enough, and for any $s \in \mathbb{R}$

$$\frac{\kappa_1}{\ln \frac{1}{\varepsilon}} \leqslant \|a^\varepsilon\|_{E^s(I)} \leqslant \frac{\kappa_2}{\ln \frac{1}{\varepsilon}}.$$

Proof:
We recall the definition of a^ε

$$a^\varepsilon = \Big(Id + (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon B\Big)^{-1} (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon v_0$$

We first show that, with respect to the norm $\| \ \|_{E^s(I)}$, we have

$$a^\varepsilon = (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon v_0 + O(\frac{1}{\ln^2 \frac{1}{\varepsilon}})$$

Indeed a usual calculus yields

$$\Big\|\Big(Id + (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon B\Big)^{-1} - \Big(Id - (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon B\Big)\Big\|_{E^s(I)} \leqslant \frac{\|(\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon B\|^2_{E^s(I)}}{1 - \|(\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon B\|_{E^s(I)}}$$

and using the estimates for the operators obtained during the proof of theorem 5.1, for ε small enough,

$$\Big\|\Big(Id + (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon B\Big)^{-1} - \Big(Id - (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon B\Big)\Big\|_{E^s(I)} \leqslant \frac{2}{\ln^2 \frac{1}{\varepsilon}}.$$

So with respect to the operator norm on $E^s(I)$, we have

$$\begin{aligned} \Big(Id + (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon B\Big)^{-1} (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} &= \Big(Id - (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon B\Big) (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \\ &\quad + O(\frac{1}{\ln^3 \frac{1}{\varepsilon}}) \\ &= (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} + O(\frac{1}{\ln^2 \frac{1}{\varepsilon}}). \end{aligned}$$

Since the family $(\pi^\varepsilon v_0)_{\varepsilon > 0}$ is bounded in $E^s(I)$, we obtain the desired result. Now we need to estimate $(\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon v_0$. There exists $\kappa_1, \kappa_2 > 0$ independant of ε such that for ε small enough, and for any $q = 0 \ldots [\frac{1}{\sqrt{\varepsilon}}]$, $\frac{\kappa_1^2}{\ln^2 \frac{1}{\varepsilon}} \leqslant \frac{1}{(\ln(\frac{\lambda_q}{\varepsilon}))^2} \leqslant \frac{\kappa_2^2}{\ln^2 \frac{1}{\varepsilon}}$, thus we obtain

$$\frac{\kappa_1^2 \|v_0\|^2_{E^s(I)}}{\ln^2 \frac{1}{\varepsilon}} \leqslant \|(\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)^{-1} \pi^\varepsilon v_0\|^2_{E^s(I)} = \sum_{q=0}^{[\frac{1}{\sqrt{\varepsilon}}]} \frac{(1+q^2)^s}{(\ln(\frac{\lambda_q}{\varepsilon}))^2} \, |\Big\langle v_0, \widetilde{P}_q \Big\rangle_{L^2(I)}|^2 \leqslant \frac{\kappa_2^2 \|v_0\|^2_{E^s(I)}}{\ln^2 \frac{1}{\varepsilon}}$$

and this concludes the proof of the lemma.

■

6 Summary on the construction of the approximate field

This section sums up the results of the preceding sectionand define the approximate field $\widetilde{u}^\varepsilon$ that is supposed to be a good *global* approximation of u^ε. First we recall the definitions we definitely adopt for a^ε, u_1^ε and U_1^ε. For ε small enough, we define a^ε as the unique solution i $\mathfrak{E}^\varepsilon$ to the equation

$$(12)_\varepsilon : \quad (\ln \frac{1}{\varepsilon^2} Id + A^\varepsilon)\, a^\varepsilon + \pi^\varepsilon B\, a^\varepsilon = \pi^\varepsilon v_0$$

Definition of the far field
We define u_1^ε as the unique function in $H^1_{loc}(\Omega \setminus J)$ satisfying

$$\begin{cases} \Delta u_1^\varepsilon + k^2 u_1^\varepsilon = 0 \quad \text{in} \quad \Omega\setminus J, \\ u_1^\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \\ u_1^\varepsilon \quad \text{outgoing}, \\ u_1^\varepsilon - \displaystyle\int_J a^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}' \in H^1_{loc}(\Omega). \end{cases}$$

Definition of the near field
For any $\nu \in [-1;+1]$, we define $\widehat{U}^\varepsilon_{1,\nu}$ as the unique function $H^1_{\text{loc}}(\omega_N(\nu))$ satisfying

$$\begin{cases} \Delta \widehat{U}^\varepsilon_{1,\nu} = 0 \quad \text{in} \quad \omega_N(\nu), \\ \widehat{U}^\varepsilon_{1,\nu} = 0 \quad \text{on} \quad \partial\omega_N(\nu), \\ \text{There exists } \alpha\in\mathbb{C} \text{ s.t } \displaystyle\lim_{\zeta\to+\infty} \| \widehat{U}^\varepsilon_{1,\nu}(\zeta;\theta) + \alpha + \frac{a^\varepsilon(\nu)}{2\pi}\ln\zeta\|_{\theta,\infty} = 0. \end{cases}$$

Now we define the approximate field $\widetilde{u}^\varepsilon$. For this purpose we introduce a cut-off function. Consider a C^∞ function $\chi:\mathbb{R}\to[0;1]$ decreasing such that $\chi(x) = 0$ for $x\geqslant 1$ and $\chi(x) = 1$ for $x \leqslant 0$. Suppose $\mathbf{x}$ has ellipsoidal coordinates $(\xi;\nu;\theta)$, then by definition

$$\chi^\varepsilon(\mathbf{x}) = \chi(\frac{\xi^2-\varepsilon-1}{\varepsilon}).$$

In ellipsoidal coordinates we have

$$(\nabla\chi^\varepsilon)(\mathbf{x}) = \frac{2\xi}{\varepsilon}\frac{\xi^2-1}{\xi^2-\nu^2}\Big(\frac{\partial\chi}{\partial\xi}\Big)(\frac{\xi^2-\varepsilon-1}{\varepsilon})$$

which leads to the inequalities

$$\|\chi^\varepsilon\|_{L^\infty(\mathcal{T}^\varepsilon)} \leqslant 1 \quad \text{et} \quad \|\Big(\frac{\xi^2-\nu^2}{\xi^2-1}\Big)^{1/2}\nabla\chi^\varepsilon\|_{L^\infty(\mathcal{T}^\varepsilon)} \leqslant \frac{4\|\chi'\|_{L^\infty(\mathbb{R})}}{\varepsilon}. \tag{14}$$

Definition of the approximate field
We define the approximate field $\widetilde{u}^\varepsilon$ by

$$\widetilde{u}^\varepsilon(\mathbf{x}) = (1-\chi^\varepsilon(\mathbf{x}))(u_0(\mathbf{x}) + u_1^\varepsilon(\mathbf{x})) + \chi^\varepsilon(\mathbf{x})U_1^\varepsilon(\mathbf{x}) \qquad \forall \mathbf{x}\in\Omega^R_\varepsilon$$

We can suppose that U_1^ε (thus $\widetilde{u}^\varepsilon$) is extended by 0 in Ω^i_ε, , so we consider $\widetilde{u}^\varepsilon$ as a function of $\mathcal{H}$. Note that $\widetilde{u}^\varepsilon = u_0 + u_1^\varepsilon$ in $\mathcal{Z}^\varepsilon_{\text{lo}}$ and $\widetilde{u}^\varepsilon = U_1^\varepsilon$ in $\mathcal{Z}^\varepsilon_{\text{pr}}$. We denote

$$\widetilde{p}^\varepsilon = \frac{\partial\widetilde{u}^\varepsilon}{\partial n}|_{\Gamma^\varepsilon} \quad \text{where } n \text{ is the normal outgoing with respect to } \Omega^i_\varepsilon.$$

$$\widetilde{f}^\varepsilon \in \mathcal{H}' \quad \text{such that} \quad \Big\langle \widetilde{f}^\varepsilon; v\Big\rangle_{\mathcal{H}',\mathcal{H}} = (\mathcal{A}\widetilde{u}^\varepsilon; v)_{\mathcal{H}} + \int_{\Gamma^\varepsilon}\widetilde{p}^\varepsilon\overline{v} + \int_{\Omega_R} f\,\overline{v}.$$

Then $(\widetilde{u}^\varepsilon;\widetilde{p}^\varepsilon)\in\mathcal{H}\times H^{-1/2}(\Gamma^\varepsilon)$ satisfies

$$(\widetilde{\mathcal{P}}^\varepsilon):\begin{cases} \displaystyle\int_{\Omega_R}\nabla\widetilde{u}^\varepsilon\cdot\nabla\overline{v} - k^2\int_{\Omega_R}\widetilde{u}^\varepsilon\overline{v} + \int_{\Gamma_R}\overline{v}T_R\widetilde{u}^\varepsilon + \int_{\Gamma^\varepsilon}\widetilde{p}^\varepsilon\overline{v} = -\int_{\Omega_R} f\overline{v} + \Big\langle\widetilde{f}^\varepsilon; v\Big\rangle_{\mathcal{H}',\mathcal{H}} \quad \forall v\in\mathcal{H}, \\ \displaystyle\int_{\Gamma^\varepsilon} q\widetilde{u}^\varepsilon = 0 \quad \forall q\in H^{-1/2}(\Gamma^\varepsilon). \end{cases}$$

7 Step 4: error estimate

Now we show theorem 7.1 that means that $\widetilde{u}^\varepsilon$ is a good approximation of u^ε. This proof will be decomposed in two sub-step: stability (1 st §) and consistancy (2 nd §).

7.1 Stability lemma

The aim of this paragraph is lemma 7.3. Gevien ε, for $f^\varepsilon \in \mathcal{H}_0^{\varepsilon'}$, we consider the problem

$$\text{Find} \quad u_0^\varepsilon \in \mathcal{H}_0^\varepsilon \quad \text{such that} \quad (\mathcal{A}_0^\varepsilon u_0^\varepsilon; v)_{\mathcal{H}} = \langle f^\varepsilon; v\rangle_{\mathcal{H}_0^{\varepsilon'},\mathcal{H}_0^\varepsilon} \qquad \forall v \in \mathcal{H}_\tau^\varepsilon. \tag{15}$$

Recall that the space $\mathcal{H}_0^\varepsilon$ was defined in §2.4.1 as the set of functions of $\mathcal{H}$ with trace on Γ^ε equal to 0.

Lemma 7.1
For $\varepsilon > 0$ small enough and for any $f^\varepsilon \in \mathcal{H}_0^{\varepsilon'}$, the problem (15) admits a unique solution. Moreover there exists $\kappa_1 > 0$ and $\kappa_2 > 0$ independant of ε such that

$$\forall \epsilon > 0 \quad \forall u \in \mathcal{H}_0^\varepsilon \quad , \qquad \kappa_1 \|u\|_{\mathcal{H}} \leqslant \|\mathcal{A}_0^\varepsilon u\|_{\mathcal{H}} \leqslant \kappa_2 \|u\|_{\mathcal{H}}.$$

Proof:
Concerning continuity, it is easy to see the existence of κ_2 independant of ε such that $\forall u \in \mathcal{H}_0^\varepsilon, \|\mathcal{A}_0^\varepsilon u\|_{\mathcal{H}} \leqslant \kappa_2 \|u\|_{\mathcal{H}}$. For the rest of the proof we shall proceed by contradiction. Suppose there exist $\varepsilon_n \to 0$ and $w_n \in \mathcal{H}_0^{\varepsilon_n}$ such that for any n, $\|w_n\|_{\mathcal{H}} = 1$ and $\|\mathcal{A}_0^{\varepsilon_n} w_n\|_{\mathcal{H}} \to 0$. We can suppose, extracting a sub-sequence if necessary, that (w_n) strongly converges in $L^2(\Omega_R)$ and weakly converges in $H^1(\Omega_R)$ toward $w_0 \in H^1(\Omega_R)$. Take $v \in H^1_{00}(\Omega_R)$ fixed. According to our initial assumption we have $(\mathcal{A}_0^{\varepsilon_n} w_n; v)_{\mathcal{H}} \leqslant \|\mathcal{A}_0^{\varepsilon_n} w_n\|_{\mathcal{H}}\, \|v\|_{H^1(\Omega_R)} \to 0$ when $n \to +\infty$. Passing to the limit by weak convergence $(\mathcal{A} w_0; v)_{\mathcal{H}} = 0$. Since v is an arbitrary function $H^1_{00}(\Omega_R \setminus J) = \{\, v \in H^1(\Omega_R) \mid v = 0 \text{ sur un voisinage de } J \,\}$ et since $H^1_{00}(\Omega_R \setminus J)$ is dense in $H^1(\Omega_R)$ (see lemma A.8 in appendix), there holds $\forall v \in H^1(\Omega_R)$, $(\mathcal{A} w_0; v)_{\mathcal{H}} = 0$. Then a classical result of uniqueness gives $w_0 = 0$. Moreover the positivity property of the Dirichlet-to-Neumann map states $\Re e\{\int_{\Gamma_R} \overline{v} T_R v\} \geqslant 0$ so

$$\|\nabla w_n\|^2_{L^2(\Omega_R)} \leqslant \|\nabla w_n\|^2_{L^2(\Omega_R)} + \Re e\{\int_{\Gamma_R} w_n\, T_R w_n\} = k^2 \|w_n\|^2_{L^2(\Omega_R)} + (\mathcal{A}_0^{\varepsilon_n} w_n; v)_{\mathcal{H}} \to 0$$

which is in contradiction with our intial assumption. In conclusion we have proved that there exists $\kappa_1 > 0$ independant of ε such that for ε small enough $\forall u \in \mathcal{H}_0^\varepsilon$, $\kappa_1 \|u\|_{\mathcal{H}} \leqslant \|\mathcal{A}_0^\varepsilon u\|_{\mathcal{H}}$. We have shown the injectivity of this operator for ε small In addition, since $\mathcal{A}_0^\varepsilon$ is of Fredholm type Fredholm alternative yields the bijectivity of the operator $\mathcal{A}_0^\varepsilon$, which means existence and uniqueness of the solution to the problem we are interested in.

■

Given $\varepsilon > 0$, for $f^\varepsilon \in \mathcal{H}'$ and $g^\varepsilon \in H^{1/2}(\Gamma^\varepsilon)$ we consider the problem

$$\left\{\begin{array}{l} \text{Find} \quad (v^\varepsilon; q^\varepsilon) \in \mathcal{H} \times H^{-1/2}(\Gamma^\varepsilon) \quad \text{such that} \\[2mm] \displaystyle\int_{\Omega_R} \nabla v^\varepsilon \cdot \nabla \overline{v} - k^2 \int_{\Omega_R} v^\varepsilon\, \overline{v} + \int_{\Gamma_R} \overline{v}\, T_R v^\varepsilon + \int_{\Gamma^\varepsilon} q^\varepsilon\, \overline{v} = \langle f^\varepsilon; v\rangle_{\mathcal{H}',\mathcal{H}} \qquad \forall v \in \mathcal{H}, \\[2mm] \displaystyle\int_{\Gamma^\varepsilon} q\, \overline{v^\varepsilon} = \langle g^\varepsilon; v\rangle_{H^{-1/2}(\Gamma^\varepsilon),H^{1/2}(\Gamma^\varepsilon)} \qquad \forall q \in H^{-1/2}(\Gamma^\varepsilon). \end{array}\right. \tag{16}$$

In order to obtain existence and uniqueness results and also continuity estimates concerning the problem (16), we will apply the theory of Brezzi and Fortin (c.f [6]).

Lemma 7.2
There exists $\kappa > 0$ independant of ε such that the trace operator τ^ε satisfies the inf-sup condition

$$\sup_{v\in\mathcal{H}} \frac{\langle q, \tau^\varepsilon v\rangle_{H^{-1/2}(\Gamma^\varepsilon), H^{1/2}(\Gamma^\varepsilon)}}{\|v\|_{\mathcal{H}}} = \sup_{v\in\mathcal{H}} \frac{\int_{\Gamma^\varepsilon} q\overline{v}}{\|v\|_{\mathcal{H}}} \geqslant \kappa\, \|q\|_{H^{-1/2}(\Gamma^\varepsilon)}$$

Proof:

Take $w \in H^{1/2}(\Gamma^\varepsilon)$ such that $\dfrac{\langle q, \tau^\varepsilon w\rangle_{H^{-1/2}(\Gamma^\varepsilon), H^{1/2}(\Gamma^\varepsilon)}}{\|w\|_{\mathcal{H}}} \geqslant \dfrac{1}{2}\|q\|_{H^{-1/2}(\Gamma^\varepsilon)}$. Then according to lemma (2.2), there exists $u \in H^1(\Omega_R)$ such that $u_{|\Gamma^\varepsilon} = w$ and $\|u\|_{H^1(\Omega_R)} \leqslant \kappa\|w\|_{H^{1/2}(\Gamma^\varepsilon)}$ with $\kappa > 0$. From this we deduce

$$\begin{aligned}\sup_{v\in\mathcal{H}} \frac{\langle q, \tau^\varepsilon v\rangle_{H^{-1/2}(\Gamma^\varepsilon), H^{1/2}(\Gamma^\varepsilon)}}{\|v\|_{\mathcal{H}}} & \geqslant \frac{\langle q, \tau^\varepsilon w\rangle_{H^{-1/2}(\Gamma^\varepsilon), H^{1/2}(\Gamma^\varepsilon)}}{\|w\|_{\mathcal{H}}} \\ & \geqslant \frac{\langle q, \tau^\varepsilon w\rangle_{H^{-1/2}(\Gamma^\varepsilon), H^{1/2}(\Gamma^\varepsilon)}}{\|w\|_{H^{1/2}(\Gamma^\varepsilon)}} \; \frac{\|w\|_{H^{1/2}(\Gamma^\varepsilon)}}{\|w\|_{H^1(\Omega_R)}} \geqslant \frac{1}{2\kappa}\, \|q\|_{H^{-1/2}(\Gamma^\varepsilon)}.\end{aligned}$$

■

Lemma 7.3
For ε small enough and for any $f^\varepsilon \in \mathcal{H}'$, $g^\varepsilon \in H^{1/2}(\Gamma^\varepsilon)$, the problem (16) admits a unique solution $(v^\varepsilon; q^\varepsilon) \in \mathcal{H} \times H^{-1/2}(\Gamma^\varepsilon)$ that satisfies in addition

$$\begin{aligned}\|v^\varepsilon\|_{\mathcal{H}} &\leqslant \frac{1}{\kappa_1}\|f^\varepsilon\|_{\mathcal{H}'} + (1 + \frac{\kappa_2}{\kappa_1})\|g^\varepsilon\|_{H^{1/2}(\Gamma^\varepsilon)} \\ \|q^\varepsilon\|_{H^{-1/2}(\Gamma^\varepsilon)} &\leqslant (1 + \frac{\kappa_2}{\kappa_1})\|f^\varepsilon\|_{\mathcal{H}'} + \kappa_2(1 + \frac{\kappa_2}{\kappa_1})\|g^\varepsilon\|_{H^{1/2}(\Gamma^\varepsilon)}\end{aligned}$$

where κ_1, κ_2 are the constants appearing in lemma 7.1.

Proof:

Uniqueness

Suppose we can find a sequence $\varepsilon_n > 0$ such that $\varepsilon_n \to 0$ when $n \to +\infty$, and such that for any $n \in \mathbb{N}$ there exists $(w_n; p_n) \in \mathcal{H} \times H^{-1/2}(\Gamma^{\varepsilon_n})$ such that

$$\begin{aligned}&\int_{\Omega_R} \nabla w_n \cdot \nabla\overline{v} - k^2 \int_{\Omega_R} w_n\, \overline{v} + \int_{\Gamma_R} \overline{v}\, T_R w_n + \int_{\Gamma^\varepsilon} p_n\, \overline{v} = 0 \qquad \forall v \in H^1(\Omega_R), \\ &\int_{\Gamma^\varepsilon} q\, \overline{w_n} = 0 \qquad \forall q \in H^{-1/2}(\Gamma^{\varepsilon_n}).\end{aligned}$$

Without restricting generality we can suppose that $\forall n,\ \|w_n\|_{\mathcal{H}} = \|w_n\|_{H^1(\Omega_R)} = 1$, and extracting a subsequence if necessary, we can suppose that there exists $w_0 \in H^1(\Omega_R)$ such that $w_n \to w_0$ when $n \to +\infty$ in a weak sense in $H^1(\Omega_R)$ and in a strong sense in $L^2(\Omega_R)$. Choose a fixed $v \in H^1_{00}(\Omega_R)$. For n large enough, we have $\int_{\Omega_R} \nabla w_n \cdot \nabla\overline{v} - k^2 \int_{\Omega_R} w_n\, \overline{v} + \int_{\Gamma_R} \overline{v}\, T_R w_n = 0$. Passing to the limit when $n \to +\infty$ we obtain $\int_{\Omega_R} \nabla w_0 \cdot \nabla\overline{v} - k^2 \int_{\Omega_R} w_0\, \overline{v} + \int_{\Gamma_R} \overline{v}\, T_R w_0 = 0$. Moreover since $H^1_{00}(\Omega_R \setminus J)$ is dense in $H^1(\Omega_R)$ (see lemma A.8) and v is arbitrary, we are led to: $\int_{\Omega_R} \nabla w_0 \cdot \nabla\overline{v} - k^2 \int_{\Omega_R} w_0\, \overline{v} + \int_{\Gamma_R} \overline{v}\, T_R w_0 = 0$, $\forall v \in H^1(\Omega_R)$. The Rellich lemma implies that $w_0 = 0$. Thus we obtain $\|w_n\|_{L^2(\Omega_R)} \to 0$. Moreover $\forall n$:

$$\|\nabla w_n\|^2_{L^2(\Omega_R)} \leqslant \|\nabla w_n\|^2_{L^2(\Omega_R)} + \Re e\{\int_{\Gamma_R} w_n\, T_R u_n\} = k^2\|w_n\|^2_{L^2(\Omega_R)} \to 0.$$

From this we deduce that $\|w_n\|_{H^1(\Omega_R)} \to 0$ which contradicts the assumption that $\|w_n\|_{H^1(\Omega_R)} = 1$. This contradiction yields the desired result.

Existence
We know that the trace operator $\tau^\varepsilon : \mathcal{H} \to H^{1/2}(\Gamma^\varepsilon)$ is onto. Moreover we have verified in lemma 7.2 an inf-sup condition for τ^ε. We can find $v^\varepsilon_{g^\varepsilon} \in \mathcal{H}$ such that $\tau^\varepsilon v^\varepsilon_{g^\varepsilon} = g^\varepsilon$ and $\|v^\varepsilon_{g^\varepsilon}\|_{\mathcal{H}} \leqslant \|g^\varepsilon\|_{H^{1/2}(\Gamma^\varepsilon)}$. According to lemma 7.1, for any ε small enough, there exists a unique $v^\varepsilon_0 \in \mathcal{H}^\varepsilon_0$ such that $(\mathcal{A}^\varepsilon_0 v^\varepsilon_0; v)_{\mathcal{H}} = (\mathcal{A} v^\varepsilon_0; v)_{\mathcal{H}} = \langle f^\varepsilon; v\rangle_{\mathcal{H}',\mathcal{H}} - (\mathcal{A} v^\varepsilon_{g^\varepsilon}; v)_{\mathcal{H}}$, $\forall v \in \mathcal{H}^\varepsilon_\tau$. Again using lemma 7.1,

$$\kappa_1 \|v^\varepsilon_0\|_{\mathcal{H}} \leqslant \|\mathcal{A}^\varepsilon_0 v^\varepsilon_0\|_{\mathcal{H}} \leqslant \|f^\varepsilon\|_{\mathcal{H}'} + \sup_{v\in\mathcal{H}^\varepsilon_0} \frac{(\mathcal{A} v^\varepsilon_{g^\varepsilon}; v)_{\mathcal{H}}}{\|v\|_{\mathcal{H}}} \leqslant \|f^\varepsilon\|_{\mathcal{H}'} + \|\mathcal{A} v^\varepsilon_{g^\varepsilon}\|_{\mathcal{H}} \leqslant \|f^\varepsilon\|_{\mathcal{H}'} + \kappa_2 \|v^\varepsilon_{g^\varepsilon}\|_{\mathcal{H}}$$

$$\|v^\varepsilon_0\|_{\mathcal{H}} \leqslant \frac{1}{\kappa_1}\|f^\varepsilon\|_{\mathcal{H}'} + \frac{\kappa_2}{\kappa_1}\|g^\varepsilon\|_{H^{1/2}(\Gamma^\varepsilon)}.$$

Note that $v^\varepsilon_0 \in \mathcal{H}^\varepsilon_0$ if and only if $v^\varepsilon_0 \in \mathcal{H}$ and $\int_{\Gamma^\varepsilon} q\,\overline{v^\varepsilon_0} = 0$, $\forall q \in H^{-1/2}(\Gamma^\varepsilon)$. Consider $v^\varepsilon = v^\varepsilon_0 + v^\varepsilon_{g^\varepsilon}$. Then

$$v^\varepsilon \in \mathcal{H} \text{ and } \int_{\Gamma^\varepsilon} q\,\overline{v^\varepsilon} = \langle q; g^\varepsilon\rangle_{H^{-1/2}(\Gamma^\varepsilon),\, H^{1/2}(\Gamma^\varepsilon)} \ , \ \forall q \in H^{-1/2}(\Gamma^\varepsilon)$$

$$\text{and} \quad \|v^\varepsilon\|_{\mathcal{H}} \leqslant \tfrac{1}{\kappa_1}\|f^\varepsilon\|_{\mathcal{H}'} + (1 + \tfrac{\kappa_2}{\kappa_1})\|g^\varepsilon\|_{H^{1/2}(\Gamma^\varepsilon)}.$$

Define $L : \mathcal{H} \to \mathbb{C}$ by the identity $L(v) = \langle f^\varepsilon; v\rangle_{\mathcal{H}',\mathcal{H}} - (\mathcal{A} v^\varepsilon; v)_{\mathcal{H}}$, $\forall v \in \mathcal{H}^\varepsilon$. For any $v \in \mathcal{H}^\varepsilon_0$, $L(v) = 0$, thus according to proposition 1.2 of [6], since the inf-sup condition is satisfied there exists $q^\varepsilon \in H^{-1/2}(\Gamma^\varepsilon)$ such that $L(v) = \langle q^\varepsilon; \tau^\varepsilon v\rangle$ and $\|q^\varepsilon\|_{H^{-1/2}(\Gamma^\varepsilon)} \leqslant \|L\|_{\mathcal{H}'}$. To sum up $(v^\varepsilon; q^\varepsilon) \in \mathcal{H} \times H^{-1/2}(\Gamma^\varepsilon)$ satisfies

$$(\mathcal{A} v^\varepsilon; v)_{\mathcal{H}} + \int_{\Gamma^\varepsilon} q^\varepsilon\,\overline{v} = \langle f^\varepsilon; v\rangle_{\mathcal{H}',\mathcal{H}} \qquad \forall v \in \mathcal{H},$$

$$\int_{\Gamma^\varepsilon} q\,\overline{v^\varepsilon} = \langle q; g^\varepsilon\rangle_{H^{-1/2}(\Gamma^\varepsilon),\, H^{1/2}(\Gamma^\varepsilon)} \qquad \forall q \in H^{-1/2}(\Gamma^\varepsilon).$$

thus is the unique solution of problem (16). Moreover the following estimate holds

$$\|q^\varepsilon\|_{H^{-1/2}(\Gamma^\varepsilon)} \leqslant \|L\|_{\mathcal{H}'} \leqslant \|f^\varepsilon\|_{\mathcal{H}'} + \kappa_2\|v^\varepsilon\|_{\mathcal{H}} \leqslant (1 + \frac{\kappa_2}{\kappa_1})\|f^\varepsilon\|_{\mathcal{H}'} + \kappa_2(1 + \frac{\kappa_2}{\kappa_1})\|g^\varepsilon\|_{H^{1/2}(\Gamma^\varepsilon)}.$$

■

7.2 Consistancy result

Theorem 7.1
Consider $(\widetilde{u}^\varepsilon, \widetilde{p}^\varepsilon) \in \mathcal{H} \times H^{-1/2}(\Gamma^\varepsilon)$ defined in section 6. There exists $\kappa > 0$ independant of ε such that

$$\|u^\varepsilon - \widetilde{u}^\varepsilon\|_{\mathcal{H}} \leqslant \kappa\,\varepsilon^{1/2}\ln 1/\varepsilon \quad \textit{et} \quad \|p^\varepsilon - \widetilde{p}^\varepsilon\|_{H^{-1/2}(\Gamma^\varepsilon)} \leqslant \kappa\,\varepsilon^{1/2}\ln 1/\varepsilon.$$

Proof:
We will decompose the proof in 5 steps:

- Express the global error: this error is itself decomposed in three parts corresponding to the other steps.
- Estimate the error for the far field: this will be very easy.
- Estimate the error for the near field: we will use a direct computation and take into account the results established in section 4.3.2 and in the last step.

- Estimate the error for the matching condition: we will also use direct computation and take into account the results of section 5.4 and last step.

- Technical results for error estimate : proof of technical results required in the preceding steps.

7.2.1 Global error

In order to estimate the global error, the stability lemma leads us to estimate $\|\widetilde{f}^\varepsilon\|_{\mathcal{H}'}$. Indeed let us consider the difference between problem $(\mathcal{P}^\varepsilon_{\mathrm{ex}})$ and $(\widetilde{\mathcal{P}}^\varepsilon)$. If we define $v^\varepsilon_{ap} = \widetilde{u}^\varepsilon - u^\varepsilon$ and $q^\varepsilon_{ap} = \widetilde{p}^\varepsilon - p^\varepsilon$, then $(v^\varepsilon_{ap}; q^\varepsilon_{ap}) \in \mathcal{H} \times H^{-1/2}(\Gamma^\varepsilon)$ satisfies

$$\left\{\begin{aligned} &\int_{\Omega_R} \nabla v^\varepsilon_{ap} \cdot \nabla \overline{v} - k^2 \int_{\Omega_R} v^\varepsilon_{ap}\overline{v} + \int_{\Gamma_R} \overline{v} T_R v^\varepsilon_{ap} + \int_{\Gamma^\varepsilon} q^\varepsilon_{ap}\overline{v} = \left\langle \widetilde{f}^\varepsilon ; v \right\rangle_{\mathcal{H}',\mathcal{H}} \qquad \forall v \in \mathcal{H},\\ &\int_{\Gamma^\varepsilon} q v^\varepsilon_{ap} = 0 \quad \forall q \in H^{-1/2}(\Gamma^\varepsilon). \end{aligned}\right.$$

Applying lemma 7.3 we obtain $\kappa > 0$ independant of ε such that

$$\|v^\varepsilon_{ap}\|_{\mathcal{H}} = \|\widetilde{u}^\varepsilon - u^\varepsilon\|_{\mathcal{H}} \leqslant \kappa\,\|\widetilde{f}^\varepsilon\|_{\mathcal{H}'} \qquad \|q^\varepsilon_{ap}\|_{H^{-1/2}(\Gamma^\varepsilon)} = \|\widetilde{p}^\varepsilon - p^\varepsilon\|_{H^{-1/2}(\Gamma^\varepsilon)} \leqslant \kappa\,\|\widetilde{f}^\varepsilon\|_{\mathcal{H}'}.$$

A direct calculus leads to

$$\|\widetilde{f}^\varepsilon\|_{\mathcal{H}'} = \sup_{v\in\mathcal{H}} \frac{\left\langle \widetilde{f}^\varepsilon ; v \right\rangle_{\mathcal{H}',\mathcal{H}}}{\|v\|_{\mathcal{H}}}$$

$$\begin{aligned} \left\langle \widetilde{f}^\varepsilon ; v \right\rangle_{\mathcal{H}',\mathcal{H}} &= \int_{\Omega_R} \nabla\widetilde{u}^\varepsilon \cdot \nabla\overline{v} - k^2 \int_{\Omega_R} \widetilde{u}^\varepsilon\overline{v} + \int_{\Gamma_R} \overline{v} T_R \widetilde{u}^\varepsilon + \int_{\Gamma^\varepsilon} \widetilde{p}^\varepsilon\overline{v} - \int_{\Omega_R} f\overline{v}\\ &= \int_{\Omega_R} f\overline{v} + \int_{\Omega^R_\varepsilon} \nabla\Big\{(1-\chi^\varepsilon)(u_0 + u^\varepsilon_1) + \chi^\varepsilon U^\varepsilon_1\Big\} \cdot \nabla\overline{v} + \int_{\Gamma^\varepsilon} \widetilde{p}^\varepsilon\overline{v}\\ &\quad -k^2 \int_{\Omega^R_\varepsilon} \Big\{(1-\chi^\varepsilon)(u_0 + u^\varepsilon_1) + \chi^\varepsilon U^\varepsilon_1\Big\}\overline{v} + \int_{\Gamma_R} \overline{v} T_R (u_0 + u^\varepsilon_1)\\ &= \int_{\Omega_R} f\overline{v} + \int_{\Omega^R_\varepsilon} \Big\{(1-\chi^\varepsilon)\nabla(u_0 + u^\varepsilon_1) + \chi^\varepsilon \nabla U^\varepsilon_1\Big\} \cdot \nabla\overline{v} - \int_{\Omega^R_\varepsilon} (u_0 + u^\varepsilon_1 - U^\varepsilon_1)\nabla\chi^\varepsilon \cdot \nabla\overline{v}\\ &\quad -k^2 \int_{\Omega^R_\varepsilon} \Big\{(1-\chi^\varepsilon)(u_0 + u^\varepsilon_1) + \chi^\varepsilon U^\varepsilon_1\Big\}\overline{v} + \int_{\Gamma_R} \overline{v} T_R (u_0 + u^\varepsilon_1) + \int_{\Gamma^\varepsilon} \widetilde{p}^\varepsilon\overline{v} \end{aligned}$$

$$\begin{aligned} &\textbf{far field error} \quad \left\{\begin{aligned} &= \int_{\Omega_R} f\overline{v} + \int_{\Omega_R} \nabla(u_0 + u^\varepsilon_1) \cdot \nabla\{(1-\chi^\varepsilon)\overline{v}\}\\ &-k^2 \int_{\Omega_R} (u_0 + u^\varepsilon_1)\{(1-\chi^\varepsilon)\overline{v}\} + \int_{\Gamma_R} \{(1-\chi^\varepsilon)\overline{v}\} T_R (u_0 + u^\varepsilon_1) \end{aligned}\right.\\ &\textbf{near field error} \quad + \int_{\Omega^R_\varepsilon} \nabla U^\varepsilon_1 \cdot \nabla\{\chi^\varepsilon\overline{v}\} - k^2 \int_{\Omega^R_\varepsilon} \chi^\varepsilon U^\varepsilon_1 \overline{v} + \int_{\Gamma^\varepsilon} \widetilde{p}^\varepsilon\overline{v}\\ &\textbf{matching error} \quad + \int_{\mathcal{T}^\varepsilon} \nabla\chi^\varepsilon \cdot \nabla(u_0 + u^\varepsilon_1 - U^\varepsilon_1)\overline{v} - \int_{\mathcal{T}^\varepsilon} (u_0 + u^\varepsilon_1 - U^\varepsilon_1)\nabla\chi^\varepsilon \cdot \nabla\overline{v} \end{aligned}$$

Looking at the problems statisfied by u_0 and u^ε_1, we see that the far field error is very easy to compute

$$\int_{\Omega^R_\varepsilon} f\overline{v} + \int_{\Omega^R_\varepsilon} \nabla(u_0 + u^\varepsilon_1) \cdot \nabla\{(1-\chi^\varepsilon)\overline{v}\} - k^2 \int_{\Omega^R_\varepsilon} (u_0 + u^\varepsilon_1)\{(1-\chi^\varepsilon)\overline{v}\} + \int_{\Gamma_R} \{(1-\chi^\varepsilon)\overline{v}\} T_R (u_0 + u^\varepsilon_1) = 0.$$

7.2.2 Near field error

We have constructed the near field in such a manner that this error is very small

$$\int_{\Omega_\varepsilon^R} \nabla U_1^\varepsilon \cdot \nabla\{\chi^\varepsilon \overline{v}\} - k^2 \int_{\Omega_\varepsilon^R} U_1^\varepsilon \chi^\varepsilon \overline{v} + \int_{\Gamma^\varepsilon} \widetilde{p}^\varepsilon \overline{v} = - \int_{\Omega_\varepsilon^R} \Delta U_1^\varepsilon \chi^\varepsilon \overline{v} - k^2 \int_{\Omega_\varepsilon^R} U_1^\varepsilon \chi^\varepsilon \overline{v}.$$

We estimate the left hand side by means of §4.3.2 and lemma 7.5,

$$\int_{\Omega_\varepsilon^R} \Delta U_1^\varepsilon \chi^\varepsilon \overline{v} \leqslant \|\Delta U_1^\varepsilon\|_{L^2(\mathcal{Z}_{\rm pr}^\varepsilon)} \|v\|_{L^2(\mathcal{Z}_{\rm pr}^\varepsilon)} \leqslant \kappa \, \|a^\varepsilon\|_{E^4(I)} \, \varepsilon^{1/2} \, \ln 1/\varepsilon \, . \, \varepsilon^{1/2} \, \ln 1/\varepsilon = \kappa \, \varepsilon \, \ln 1/\varepsilon,$$

$$\int_{\Omega_\varepsilon^R} \chi^\varepsilon U_1^\varepsilon \overline{v} \leqslant \|U_1^\varepsilon\|_{L^2(\mathcal{Z}_{\rm pr}^\varepsilon)} \|v\|_{L^2(\mathcal{Z}_{\rm pr}^\varepsilon)} \leqslant \kappa \, \varepsilon^{1/2} \, \|V_0\|_{L^\infty(\mathcal{Z}_{\rm pr}^\varepsilon)} \|a^\varepsilon\|_{L^2(I)} \|v\|_{L^2(\mathcal{Z}_{\rm pr}^\varepsilon)} \leqslant \kappa \, \varepsilon \ln^2 1/\varepsilon.$$

7.2.3 Matching error

Now we tackle the matching error. We have chosen a^ε for this error to be small

$$\begin{aligned}
\int_{\Omega_\varepsilon^R} (u_0 + u_1^\varepsilon - U_1^\varepsilon) \nabla \chi^\varepsilon \cdot \nabla \overline{v} &= \int_{\mathcal{T}^\varepsilon} (u_0 + u_1^\varepsilon - U_1^\varepsilon) \frac{\partial \chi^\varepsilon}{\partial \xi} \frac{\partial \overline{v}}{\partial \xi} (\xi^2 - 1) \, d\xi d\nu d\theta \\
&\leqslant \Big(\int_{\mathcal{T}^\varepsilon} \Big| (u_0 + u_1^\varepsilon - U_1^\varepsilon) \frac{\partial \chi^\varepsilon}{\partial \xi} \Big|^2 (\xi^2 - 1) \, d\xi d\nu d\theta \Big)^{1/2} \, \|v\|_{\mathcal{H}} \\
&\leqslant \frac{\kappa}{\varepsilon} \Big(\int_{\mathcal{T}^\varepsilon} \Big| u_0 + u_1^\varepsilon - U_1^\varepsilon \Big|^2 (\xi^2 - 1) \, d\xi d\nu d\theta \Big)^{1/2} \, \|v\|_{\mathcal{H}} \\
&\leqslant \frac{\kappa}{\varepsilon^{1/2}} \Big(\int_{\xi = \sqrt{1+\varepsilon}}^{\sqrt{1+2\varepsilon}} (\xi^2 - 1) \, d\xi \Big)^{1/2} \, \Big\| u_0 + u_1^\varepsilon - U_1^\varepsilon \Big\|_{L_\xi^\infty \otimes L_\nu^2(\mathcal{T}^\varepsilon)} \, \|v\|_{\mathcal{H}} \\
&\leqslant \kappa \, \varepsilon^{1/2} \, \sqrt{\ln 1/\varepsilon} \, \|v\|_{\mathcal{H}}.
\end{aligned}$$

For the second term, using corollary 7.1 we also have

$$\begin{aligned}
\int_{\Omega_\varepsilon^R} \nabla \chi^\varepsilon \cdot \nabla (u_0 + u_1^\varepsilon - U_1^\varepsilon) \overline{v} &= \int_{\mathcal{T}^\varepsilon} \overline{v} \frac{\partial \chi^\varepsilon}{\partial \xi} \frac{\partial}{\partial \xi} (u_0 + u_1^\varepsilon - U_1^\varepsilon) \, (\xi^2 - 1) \, d\xi \, d\nu \, d\theta \\
&\leqslant \Big(\int_{\mathcal{T}^\varepsilon} \Big| \frac{\partial}{\partial \xi} (u_0 + u_1^\varepsilon - U_1^\varepsilon) \Big|^2 (\xi^2 - 1) \, d\xi d\nu d\theta \Big)^{1/2} \, \|v \nabla \chi^\varepsilon\|_{L^2(\mathcal{T}^\varepsilon)} \\
&\leqslant \kappa \, \ln 1/\varepsilon \, \Big(\int_{\xi = \sqrt{1+\varepsilon}}^{\sqrt{1+2\varepsilon}} (\xi^2 - 1) \, d\xi \Big)^{1/2} \, \Big\| \frac{\partial}{\partial \xi} \{ u_0 + u_1^\varepsilon - U_1^\varepsilon \} \Big\|_{L_\xi^\infty \otimes L_\nu^2(\mathcal{T}^\varepsilon)} \, \|v\|_{\mathcal{H}} \\
&\leqslant \kappa \, \varepsilon^{1/2} \, \ln 1/\varepsilon \, \|v\|_{\mathcal{H}}.
\end{aligned}$$

7.2.4 Estimation results

Lemma 7.4
For any $\varepsilon > 0$ and for any $v \in \mathcal{H}$, there exists $\kappa > 0$ such that $\|\chi^\varepsilon v\|_{\mathcal{H}} \leqslant \kappa \, \|v\|_{\mathcal{H}} \, \ln 1/\varepsilon$.

Proof:
According to the density of $C^\infty(\Omega_R)$ in $\mathcal{H}$ it is sufficient to show that v is very smooth. Take $\mathcal{O}$ and ξ_0 like in §2.3.3. We can assume that ξ_0 is sufficiently close to 1, then suppose that $\Omega_\varepsilon^i \subset \mathcal{O} \subset \Omega_R$ for ε small enough. Take a C^∞ cut-off function $\widetilde{\chi}$ such that $\widetilde{\chi}(\mathbf{x}) = 1$ if $\xi < \xi_0/2$ and $\widetilde{\chi}(\mathbf{x}) = 0$ if $\xi > \xi_0$. since the $v \mapsto \widetilde{\chi} v$ maps continuously $\mathcal{H}$ into itself, we can restrict ourselves

to functions $v \in \mathcal{H}$ such that $v(\mathbf{x}) = 0$ if $\xi > \xi_0$. Using the estimates (14), we can bound $\|\chi^{\varepsilon} v\|_{\mathcal{H}}$ in the following manner.

$$\|\chi^{\varepsilon} v\|_{\mathcal{H}}^2 = \|\chi^{\varepsilon} v\|_{L^2(\Omega_R)}^2 + \|\nabla(\chi^{\varepsilon} v)\|_{L^2(\Omega_R)}^2 \leqslant \|v\|_{L^2(\Omega_R)}^2 + 2\|\nabla v\|_{L^2(\Omega_R)}^2 + 2\frac{\kappa_1}{\varepsilon^2}\| \, v\left(\frac{\xi^2-1}{\xi^2-\nu^2}\right)^{1/2}\|_{L^2(\mathcal{T}^{\varepsilon})}^2.$$

As a consequence it is sufficient to suitably estimate the right hand side in the above identity.

$$\begin{aligned}
|v(\xi;\nu;\theta)| &= \Big|\int_{\xi}^{\xi_0} \frac{\partial v}{\partial \xi}(\tau;\nu;\theta) d\tau d\nu d\theta\Big| \\
&\leqslant \Big(\int_{\xi}^{\xi_0} \Big|\frac{\partial v}{\partial \xi}(\tau;\nu;\theta)\Big|^2 (\tau^2-1) d\tau d\nu d\theta\Big)^{1/2} \int_{\xi}^{\xi_0} \frac{d\tau}{\tau^2-1} \\
&\leqslant \kappa_2 \ln\Big(\frac{\xi+1}{\xi-1}\Big) \Big(\int_{\xi}^{\xi_0} \Big|\frac{\partial v}{\partial \xi}(\tau;\nu;\theta)\Big|^2 (\tau^2-1) d\tau d\nu d\theta\Big)^{1/2}.
\end{aligned}$$

$$\begin{aligned}
\| \, v\left(\frac{\xi^2-1}{\xi^2-\nu^2}\right)^{1/2}\|_{L^2(\mathcal{T}^{\varepsilon})}^2 &= \int_{\theta=0}^{2\pi}\int_{\nu=-1}^{+1}\int_{\xi=\sqrt{1+\varepsilon}}^{\sqrt{1+2\varepsilon}} |v(\xi;\nu;\theta)|^2 (\xi^2-1) d\xi d\nu d\theta \\
&\leqslant \kappa_2 \, \|v\|_{\mathcal{H}} \int_{\xi=\sqrt{1+\varepsilon}}^{\sqrt{1+2\varepsilon}} (\xi^2-1)\ln^2\Big(\frac{\xi+1}{\xi-1}\Big) d\xi.
\end{aligned}$$

A simple computation shows that there exists $\kappa_3 > 0$ such that

$$\int_{\xi=\sqrt{1+\varepsilon}}^{\sqrt{1+2\varepsilon}} (\xi^2-1)\ln^2\Big(\frac{\xi+1}{\xi-1}\Big) d\xi \leqslant \kappa_3 \, \varepsilon^2 \, \ln^2 1/\varepsilon$$

which concludes the proof.

■

We can use a part of the preceding proof to obtain the next corollary

Corollary 7.1
For any $\varepsilon > 0$ and for any $v \in \mathcal{H}$,there exists $\kappa > 0$ such that $\|v\nabla\chi^{\varepsilon}\|_{L^2(\mathcal{T}^{\varepsilon})} \leqslant \kappa \, \|v\|_{\mathcal{H}} \, \ln 1/\varepsilon$

We also prove another technical lemma

Lemma 7.5
For any $\varepsilon > 0$ and for any $v \in \mathcal{H}$, there exists $\kappa > 0$ such that $\|\chi^{\varepsilon} v\|_{L^2(\Omega_R)} \leqslant \kappa \, \|v\|_{\mathcal{H}} \, \varepsilon^{1/2} \, \ln 1/\varepsilon$

Proof:
Like in the previous lemma , it is sufficient to prove this result for C^{∞} functions v such that $v(\mathbf{x}) = 0$ when $\xi > \xi_0$ for ξ_0 fixed. Using the proof of lemma 7.4,

$$\begin{aligned}
\|v\|_{L^2(\mathcal{Z}_{\mathrm{pr}}^{\varepsilon})}^2 &= \int_{\theta=0}^{2\pi}\int_{\nu=-1}^{+1}\int_{\xi=\sqrt{1+\varepsilon}}^{\sqrt{1+2\varepsilon}} |v(\xi;\nu;\theta)|^2 (\xi^2-\nu^2) d\xi d\nu d\theta \\
&\leqslant \kappa \, \|v\|_{\mathcal{H}}^2 \int_{\xi=\sqrt{1+\varepsilon}}^{\sqrt{1+2\varepsilon}} (\xi^2-\nu^2)\ln\Big(\frac{\xi+1}{\xi-1}\Big) d\xi.
\end{aligned}$$

Again we obatin the existence of a $\kappa > 0$ such that

$$\int_{\xi=\sqrt{1+\varepsilon}}^{\sqrt{1+2\varepsilon}} (\xi^2-\nu^2)\ln^2\Big(\frac{\xi+1}{\xi-1}\Big) d\xi \leqslant \kappa \, \varepsilon \, \ln^2 1/\varepsilon.$$

■

7.2.5 Conclusion

Gathering the results above, we obtain the existence of $\kappa > 0$ independant of ε such that

$$\|\widetilde{f}^\varepsilon\|_{\mathcal{H}'} \leqslant \kappa\,\varepsilon^{1/2}\,\ln 1/\varepsilon.$$

this concludes the proof of theorem 7.1.

■

8 The wire model

8.1 Averaging on sections of the wire

With $\widetilde{u}^\varepsilon$ we constructed a good approximation of u^ε. Such an approximation is interesting because it is easier to ibatin information about $\widetilde{u}^\varepsilon$. In particular close to Γ^ε, behaves like the solution of an electrostatic problem. We will use this information. We introduce a new funstional setting fro the functions that "live" on the wire Γ^ε. Let us recall that we have supposed that the surface measure on Γ^ε is $\gamma^\varepsilon(\theta,\nu)d\theta d\nu$. If $u \in C^\infty(\Gamma^\varepsilon)$ we define the function $\mu^\varepsilon u$ by

$$(\mu^\varepsilon u)(\nu) = \frac{\displaystyle\int_0^{2\pi} u \circ (\phi_{\mathrm{el}}^{\Gamma^\varepsilon})^{-1}(\nu;\theta)\,\frac{\partial V}{\partial n} \circ (\phi_{\mathrm{el}}^{\Gamma^\varepsilon})^{-1}(\nu;\theta)\,\gamma^\varepsilon(\nu;\theta)\,d\theta}{\displaystyle\int_0^{2\pi} \frac{\partial V}{\partial n} \circ (\phi_{\mathrm{el}}^{\Gamma^\varepsilon})^{-1}(\nu;\theta)\;\gamma^\varepsilon(\nu;\theta)\,d\theta}.$$

The above expression is well defined because according to lemma 4.3 the denominator is different from 0 for ε small enough. Note also that for $u \in C^\infty(\Gamma^\varepsilon)$ such that $\frac{\partial u}{\partial \theta} = 0$ (u does not depend on the variable θ), we have $\mu^\varepsilon(u) = u$. Moreover if we suppose that $u \in C^\infty(\Gamma^\varepsilon)$ then $\mu^\varepsilon(u) \in L^2(I)$. There exists $\kappa > 0$ independant of $u \in C^\infty(\Gamma^\varepsilon)$ (but also independant of ε) such that $\|\mu^\varepsilon u\|_{L^2(I)} \leqslant \kappa \|u\|_{H^{1/2}(\Gamma^\varepsilon)}$ (this is a simple consequence of Cauchy-Schwartz inequality). Using density we define a continuous map $\mu^\varepsilon : H^{1/2}(\Gamma^\varepsilon) \to L^2(I)$. We denote its range $\mathcal{M}^\varepsilon = Im\,\mu^\varepsilon$ and its kernel $H^{1/2}_\mu(\Gamma^\varepsilon) = Ker\,\mu^\varepsilon$. We have straightforwardly $H^{1/2}_\mu(\Gamma^\varepsilon) \subset Ker\,\widetilde{p}^\varepsilon$ (i.e. for $v \in H^{1/2}_\mu(\Gamma^\varepsilon)$, $\langle \widetilde{p}^\varepsilon; v\rangle = 0$). Thus μ^ε induces an isomorphism $\overline{\mu}^\varepsilon : H^{1/2}(\Gamma^\varepsilon)/H^{1/2}_\mu(\Gamma^\varepsilon) \to \mathcal{M}^\varepsilon$. Considering the quotient topology on $H^{1/2}(\Gamma^\varepsilon)/H^{1/2}_\mu(\Gamma^\varepsilon)$, we can supply $\mathcal{M}^\varepsilon$ with the unique topology for which $\overline{\mu}^\varepsilon$ is an homeomorphism. Then we have

$$\|u\|_{\mathcal{M}^\varepsilon} = \inf_{v_0 \in H^{1/2}_\mu(\Gamma^\varepsilon)} \|v + v_0\|_{H^{1/2}(\Gamma^\varepsilon)} \quad \text{for } u \in \mathcal{M}^\varepsilon\,,\ v \in H^{1/2}(\Gamma^\varepsilon) \ \text{ with } \ \mu^\varepsilon v = u.$$

Denote $\mathcal{M}^{\varepsilon'}$ the topological dual $\mathcal{M}^\varepsilon$. A natural question consist in asking whether a link exists between $\mathcal{M}^\varepsilon$ and the spaces $E^n(I)$ both used for the description of functions on the wire. The answer is given in the next lemma.

Theorem 8.1

$$\mathcal{M}^\varepsilon = E^{1/2}(I) \quad \textit{and} \quad \mathcal{M}^{\varepsilon'} = E^{-1/2}(I).$$

Proof:

According to lemma 2.1, the map $(\phi_{\mathrm{el}}^{\Gamma^\varepsilon})^{-1} \circ \phi_{\mathrm{sp}}^{S^2}$ is C^1- difféomorphism and it induces the map $\big((\phi_{\mathrm{el}}^{\Gamma^\varepsilon})^{-1} \circ \phi_{\mathrm{sp}}^{S^2}\big)^*$ defined by $\big((\phi_{\mathrm{el}}^{\Gamma^\varepsilon})^{-1} \circ \phi_{\mathrm{sp}}^{S^2}\big)^* u = u \circ (\phi_{\mathrm{el}}^{\Gamma^\varepsilon})^{-1} \circ \phi_{\mathrm{sp}}^{S^2}$ when $u \in L^2(\Gamma^\varepsilon)$. $\big((\phi_{\mathrm{el}}^{\Gamma^\varepsilon})^{-1} \circ \phi_{\mathrm{sp}}^{S^2}\big)^*$ is a continuous isomorphism from $L^2(\Gamma^\varepsilon)$ onto $L^2(S^2)$, and from $H^1(\Gamma^\varepsilon)$ onto $H^1(S^2)$. According to classical interpolation result, we obtain that $\big((\phi_{\mathrm{el}}^{\Gamma^\varepsilon})^{-1} \circ \phi_{\mathrm{sp}}^{S^2}\big)^*$ is also an isomorphism from $H^{1/2}(\Gamma^\varepsilon)$ onto $H^{1/2}(S^2)$. The function $\gamma^\varepsilon \frac{\partial V}{\partial n}$ is C^∞ on Γ^ε so if $u \in H^{1/2}(\Gamma^\varepsilon)$ then $u\,\gamma^\varepsilon \frac{\partial V}{\partial n} \in H^{1/2}(\Gamma^\varepsilon)$. Moreover according to classical results related to the decomposition in spherical harmonics, if $u \in H^{1/2}(S^2)$ then $\nu \mapsto \int_0^{2\pi} u \circ (\phi_{sp}^{\Gamma^\varepsilon})^{-1}(\nu,\theta)\,d\theta$ belongs to $E^{1/2}(I)$. For any $u \in H^{1/2}(\Gamma^\varepsilon)$, $\mu^\varepsilon(u)(\nu) =$

$\int_0^{2\pi}(u\,\frac{\partial V}{\partial n})\circ(\phi_{\rm el}^{\Gamma^\varepsilon})^{-1}(\nu,\theta)\gamma^\varepsilon(\nu,\theta)\,d\theta = \int_0^{2\pi}(u\,\frac{\partial V}{\partial n})\circ\big((\phi_{\rm el}^{\Gamma^\varepsilon})^{-1}\circ\phi_{\rm sp}^{S^2}\big)\circ(\phi_{\rm sp}^{S^2})^{-1}(\nu,\theta)\gamma^\varepsilon(\nu,\theta)\,d\theta$ so $\mu^\varepsilon(u)\in E^{1/2}(I)$.

To sum up We conclude that $\mu^\varepsilon(H^{1/2}(\Gamma^\varepsilon)) = \mathcal{M}^\varepsilon \subset E^{1/2}(I)$

Now we would like to prove the inverse inclusion. Consider the injective map $\sigma^\varepsilon = \big((\phi_{\rm el}^{\Gamma^\varepsilon})^{-1}\big)^* : E^{1/2}(I) \to H^{1/2}(\Gamma^\varepsilon)$, indeed according to equation (4) of paragraph 2.4.3, if $u \in E^{1/2}(I)$, then $\big((\phi_{\rm el}^{\Gamma^\varepsilon})^{-1}\big)^* u \in H^{1/2}(\Gamma^\varepsilon)$. Routine computations show that $\mu^\varepsilon\circ\sigma^\varepsilon = Id_{E^{1/2}(I)}$ hence $E^{1/2}(I)\subset\mathcal{M}^\varepsilon = Im\,\mu^\varepsilon$.

Conclusion $\quad E^{1/2}(I)\subset\mathcal{M}^\varepsilon$ so $E^{1/2}(I) = \mathcal{M}^\varepsilon$

Note that we have two available topologies on $E^{1/2}(I) = \mathcal{M}^\varepsilon$: the one defined for $E^{1/2}(I)$ in paragraph 5.2, and the one defined at the beginning of this currrent paragraph for $\mathcal{M}^\varepsilon$. We will prove that these two topologies coincide which will imply $\mathcal{M}^{\varepsilon'} = E^{-1/2}(I)$. In order to prove this it is sufficient to prove that $Id_{E^{1/2}(I)}$ is continuous from $(E^{1/2}(I), \|\ \|_{E^{1/2}(I)})$ into $(\mathcal{M}^\varepsilon, \|\ \|_{\mathcal{M}^\varepsilon})$. Note that $Id_{E^{1/2}(I)} = \mu^\varepsilon\circ\sigma^\varepsilon$ and μ^ε are continuous from $H^{1/2}(\Gamma^\varepsilon)$ in $(\mathcal{M}^\varepsilon, \|\ \|_{\mathcal{M}^\varepsilon})$. So it only remains to prove that σ^ε is continuous from $(E^{1/2}(I), \|\ \|_{E^{1/2}(I)})$ into $H^{1/2}(\Gamma^\varepsilon)$. This is the aim of next lemma.

Conclusion: $\quad \mathcal{M}^{\varepsilon'} = E^{-1/2}(I)$

This ends the proof of theorem 8.1.

■

Remark The preceding construction showed that the map σ^ε satisfies the following remarkable property,

$$\mu^\varepsilon\circ\sigma^\varepsilon = Id_{E^{1/2}(I)} \quad \text{and also} \quad {}^t\sigma^\varepsilon\circ{}^t\mu^\varepsilon = Id_{H^{-1/2}(\Gamma^\varepsilon)}.$$

From now on we will always use the norm defined for $E^{1/2}(I)$ in paragraph 5.2 that is denoted $\|\ \|_{E^{1/2}(I)}$. We will now give some continuity estimates for the new operators we have just introduced. One interesting property is that their norm is bounded uniformly with respect to ε.

Lemma 8.1
There exists $\kappa > 0$ independant of ε, and $\varepsilon_0 > 0$ such that $\forall\varepsilon\in]0,\varepsilon_0[$,

$$\begin{array}{lll}
(i) & \forall u\in H^{1/2}(\Gamma^\varepsilon), & \|\mu^\varepsilon(u)\|_{E^{1/2}(I)} \leqslant \kappa\|u\|_{H^{1/2}(\Gamma^\varepsilon)} \\
(ii) & \forall u\in E^{1/2}(I), & \|u\|_{E^{1/2}(I)} = \|\sigma^\varepsilon(u)\|_{H^{1/2}(\Gamma^\varepsilon)}, \\
(iii) & \forall u\in H^{-1/2}(\Gamma^\varepsilon), & \|{}^t\sigma^\varepsilon(u)\|_{E^{-1/2}(I)} \leqslant \|u\|_{H^{-1/2}(\Gamma^\varepsilon)}, \\
(iv) & \forall u\in E^{-1/2}(I), & \|u\|_{E^{-1/2}(I)} \leqslant \|{}^t\mu^\varepsilon(u)\|_{H^{-1/2}(\Gamma^\varepsilon)} \leqslant \kappa\|u\|_{E^{-1/2}(I)}.
\end{array}$$

Proof:
In order to make computation more compact we introduce a new notation,

$$\alpha_\varepsilon = \int_0^{2\pi}\frac{\partial V}{\partial n}\gamma^\varepsilon(\nu,\theta)\,d\theta.$$

We recall that according to lemma 4.3, $\lim_{\varepsilon\to 0}\alpha_\varepsilon = 2\pi$, so it is authorized to divide by this number.

(i) In order to prove this point, we begin by explicitly write $\|\mu^\varepsilon(u)\|_{E^{1/2}(I)}$,

$$\|\mu^\varepsilon(u)\|^2_{E^{1/2}(I)} = \frac{1}{\alpha_\varepsilon^2}\sum_{n=0}^{+\infty}(1+n^2)^{1/2}\Big(\int_{-1}^{+1}\int_0^{2\pi}\big((\phi_{\rm el}^{\Gamma^\varepsilon})^{-1}\big)^*\big(u\,\frac{\partial V}{\partial n}\gamma^\varepsilon\big)(\nu,\theta)\,\widetilde{P}_n(\nu)\,d\theta\,d\nu\Big)^2.$$

Moreover notice that this expression like a $H^{1/2}$ norm on the sphere S^2, indeed

$$\begin{aligned}\|\big((\phi_{\rm el}^{\Gamma^\varepsilon})^{-1}\circ\phi_{\rm sp}^{S^2}\big)^*\big(u\,\frac{\partial V}{\partial n}\gamma^\varepsilon\big)\|^2_{H^{1/2}(S^2)}&\\ = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^{1/2}\Big(\int_{-1}^{+1}\int_0^{2\pi}\big((\phi_{\rm el}^{\Gamma^\varepsilon})^{-1}\big)^*\big(u\,\frac{\partial V}{\partial n}\gamma^\varepsilon\big)(\nu,\theta)\,\widetilde{P}_n(\nu)\,d\theta\,d\nu\Big)^2.&\end{aligned}$$

Thus by taking π as a lower bound for α_ε (for ε small enough), we obtain

$$\|\mu^\varepsilon(u)\|^2_{E^{1/2}(I)} \leqslant \frac{1}{\pi}\|\big((\phi_{\rm el}^{\Gamma^\varepsilon})^{-1}\circ\phi_{\rm sp}^{S^2}\big)^*\big(u\,\frac{\partial V}{\partial n}\gamma^\varepsilon\big)\|^2_{H^{1/2}(S^2)}.$$

Since $\big((\phi_{\rm el}^{\Gamma^\varepsilon})^{-1}\circ\phi_{\rm sp}^{S^2}\big)^*\big(\frac{\partial V}{\partial n}\gamma^\varepsilon\big)$ is very smooth, interpolation gives the existence of $\kappa>0$ independant of ε such that

$$\|\big((\phi_{\rm el}^{\Gamma^\varepsilon})^{-1}\circ\phi_{\rm sp}^{S^2}\big)^*\big(u\,\frac{\partial V}{\partial n}\gamma^\varepsilon\big)\|^2_{H^{1/2}(S^2)} \;\leqslant\; \kappa\,\|\big((\phi_{\rm el}^{\Gamma^\varepsilon})^{-1}\circ\phi_{\rm sp}^{S^2}\big)^* u\|^2_{H^{1/2}(S^2)} = \kappa\,\|u\|^2_{H^{1/2}(\Gamma^\varepsilon)}.$$

The last three inequalities are more easy to prove.

(ii) This inequality is a straightforward consequence of the definition of the norm $\|\;\|_{H^{1/2}(\Gamma^\varepsilon)}$ and the defintion of σ^ε.

(iii) This inequality is due to

$$\begin{aligned}\|\,{}^t\sigma^\varepsilon(p)\|_{E^{-1/2}(I)} &= \sup_{v\in E^{1/2}(I)}\frac{\langle\,{}^t\sigma^\varepsilon(p),v\rangle_{E^{-1/2}(I),E^{1/2}(I)}}{\|v\|_{E^{1/2}(I)}}\\ &= \sup_{v\in E^{1/2}(I)}\frac{\langle p,\sigma^\varepsilon(v)\rangle_{H^{-1/2}(\Gamma^\varepsilon),H^{1/2}(\Gamma^\varepsilon)}}{\|\sigma^\varepsilon(v)\|_{H^{1/2}(\Gamma^\varepsilon)}}\\ &\leqslant \sup_{v\in H^{1/2}(\Gamma^\varepsilon)}\frac{\langle p,v\rangle_{H^{-1/2}(\Gamma^\varepsilon),H^{1/2}(\Gamma^\varepsilon)}}{\|v\|_{H^{1/2}(\Gamma^\varepsilon)}} \qquad = \|p\|_{H^{-1/2}(\Gamma^\varepsilon)}.\end{aligned}$$

(iv) This inequality is partially a consequence of (i). For the rest of the proof,

$$\begin{aligned}\|p\|_{E^{-1/2}(I)} &= \sup_{v\in E^{1/2}(I)}\frac{\langle p,v\rangle_{E^{-1/2}(I),E^{1/2}(I)}}{\|v\|_{E^{1/2}(I)}} = \sup_{v\in E^{1/2}(I)}\frac{\langle p,\mu^\varepsilon\circ\sigma^\varepsilon(v)\rangle_{E^{-1/2}(I),E^{1/2}(I)}}{\|\sigma^\varepsilon(v)\|_{H^{1/2}(\Gamma^\varepsilon)}}\\ &= \sup_{v\in E^{1/2}(I)}\frac{\langle\,{}^t\mu^\varepsilon(p),\sigma^\varepsilon(v)\rangle_{H^{-1/2}(\Gamma^\varepsilon),H^{1/2}(\Gamma^\varepsilon)}}{\|\sigma^\varepsilon(v)\|_{H^{1/2}(\Gamma^\varepsilon)}}\\ &\leqslant \sup_{v\in H^{1/2}(\Gamma^\varepsilon)}\frac{\langle\,{}^t\mu^\varepsilon(p),v\rangle_{H^{-1/2}(\Gamma^\varepsilon),H^{1/2}(\Gamma^\varepsilon)}}{\|v\|_{H^{1/2}(\Gamma^\varepsilon)}} = \|\,{}^t\mu^\varepsilon(p)\|_{H^{-1/2}(\Gamma^\varepsilon)}.\end{aligned}$$

Moreover using (i) and the fact that μ^ε is onto,

$$\begin{aligned}
\|\,{}^t\mu^\varepsilon(p)\|_{H^{-1/2}(\Gamma^\varepsilon)} &= \sup_{v\in H^{1/2}(\Gamma^\varepsilon)} \frac{\langle\,{}^t\mu^\varepsilon(p), v\rangle_{H^{-1/2}(\Gamma^\varepsilon),H^{1/2}(\Gamma^\varepsilon)}}{\|v\|_{H^{1/2}(\Gamma^\varepsilon)}} \\
&= \sup_{v\in H^{1/2}(\Gamma^\varepsilon)} \frac{\langle p, \mu^\varepsilon(v)\rangle_{E^{-1/2}(I),E^{1/2}(I)}}{\|\mu^\varepsilon(v)\|_{E^{1/2}(I)}} \frac{\|\mu^\varepsilon(v)\|_{E^{1/2}(I)}}{\|v\|_{H^{1/2}(\Gamma^\varepsilon)}} \\
&\leqslant \kappa \sup_{v\in E^{1/2}(I)} \frac{\langle p, v\rangle_{E^{-1/2}(I),E^{1/2}(I)}}{\|v\|_{E^{1/2}(I)}} = \|p\|_{E^{-1/2}(I)}.
\end{aligned}$$

■

8.2 Estimation for the Lagrange multiplier

We will now use the result of theorem 7.1. The meaning of this result is clear concerning de u^ε and $\widetilde{u}^\varepsilon$. Indeed $\widetilde{u}^\varepsilon \to u_0 \neq 0$ so we have

$$u^\varepsilon = \widetilde{u}^\varepsilon + O(\varepsilon^{1/2}) = \widetilde{u}^\varepsilon + o(\widetilde{u}^\varepsilon) \qquad \text{for the norm } \|\ \|_{H^1(\Omega_R)}.$$

However concerning p^ε, $\widetilde{p}^\varepsilon$ we don't know their behavior at first order so we cannot directly conclude in the same manner. However our discussion concerning $E^{-1/2}(I)$ will allow to remedy to this situation. Indeed note that

$$\widetilde{p}^\varepsilon = {}^t\mu^\varepsilon(-\frac{\alpha_\varepsilon}{2\pi}\, a^\varepsilon) \quad \text{with} \quad \alpha_\varepsilon = \int_0^{2\pi} \frac{\partial V}{\partial n} \gamma^\varepsilon(\nu,\theta)\, d\theta \underset{\varepsilon\to 0}{\to} 2\pi \tag{17}$$

and applying lemmas 5.4 and 8.1, we can find four constants $\kappa_i > 0$, $i = 1 \ldots 4$ independant of ε, satisfying

$$\frac{\kappa_1}{\ln\frac{1}{\varepsilon}} \leqslant \kappa_2\, \|a^\varepsilon\|_{E^{-1/2}(I)} \leqslant \|\widetilde{p}^\varepsilon\|_{H^{-1/2}(\Gamma^\varepsilon)} = \|\,{}^t\mu^\varepsilon(-\frac{\alpha_\varepsilon}{2\pi}\, a^\varepsilon)\|_{H^{-1/2}(\Gamma^\varepsilon)} \leqslant \kappa_3\, \|a^\varepsilon\|_{E^{-1/2}(I)} \leqslant \frac{\kappa_4}{\ln\frac{1}{\varepsilon}}\,,$$

where $\widetilde{p}^\varepsilon = O(\frac{1}{\ln 1/\varepsilon})$ in the norm $\|\ \|_{H^{-1/2}(\Gamma^\varepsilon)}$. Thus we finally obtain the desired conclusion

$$p^\varepsilon = \widetilde{p}^\varepsilon + O(\varepsilon^{1/2}) = \widetilde{p}^\varepsilon + o(\widetilde{p}^\varepsilon) \qquad \text{for the norm } \|\ \|_{H^{-1/2}(\Gamma^\varepsilon)}.$$

We will apply this result for the resolution of the Helmholtz problem with Dirichlet boundary condition by means of single layer integral equation on the wire. Let us first recall that $(u^\varepsilon, p^\varepsilon)$ is defined as the unique pair chosen in $H^1(\Omega_R) \times H^{-1/2}(\Gamma^\varepsilon)$ solution of the problem

$$(\mathcal{P}^\varepsilon): \left\{ \begin{array}{l} \displaystyle\int_{\Omega_R} \nabla u^\varepsilon \cdot \nabla \overline{v} - k^2 \int_{\Omega_R} u^\varepsilon \overline{v} + \int_{\Gamma_R} \overline{v} T_R u^\varepsilon + \int_{\Gamma^\varepsilon} p^\varepsilon \overline{v} = -\int_{\Omega_R} f\overline{v} \quad \forall v \in \mathcal{H}\,, \\ \displaystyle\int_{\Gamma^\varepsilon} q u^\varepsilon = 0 \quad \forall q \in H^{-1/2}(\Gamma^\varepsilon). \end{array} \right.$$

Let us substract to these equations the equations satisfied by u_0, we obtain

$$\left\{ \begin{array}{l} \displaystyle\int_{\Omega_R} \nabla(u^\varepsilon - u_0) \cdot \nabla\overline{v} - k^2 \int_{\Omega_R} (u^\varepsilon - u_0)\overline{v} + \int_{\Gamma_R} \overline{v} T_R(u^\varepsilon - u_0) + \int_{\Gamma^\varepsilon} p^\varepsilon \overline{v} = 0 \quad \forall v \in \mathcal{H}\,, \\ \displaystyle\int_{\Gamma^\varepsilon} q(u^\varepsilon - u_0) = -\int_{\Gamma^\varepsilon} q u_0 \quad \forall q \in H^{-1/2}(\Gamma^\varepsilon)\,. \end{array} \right.$$

Successively choosing $v \in \mathcal{D}(\Omega^i_\varepsilon)$ and $v \in \mathcal{D}(\Omega^R_\varepsilon)$ arbitrary, we obtain that $\Delta(u^\varepsilon - u_0) + k^2(u^\varepsilon - u_0) = 0$ in Ω^i_ε and in Ω^R_ε and $u^\varepsilon - u_0$ is outgoing. Now choosing $v \in \mathcal{D}(\Omega_R)$ arbitrarily and applying Green's formula, we obtain

$$[\frac{\partial(u^\varepsilon - u_0)}{\partial n}]_{|\Gamma^\varepsilon} = p^\varepsilon$$

where $[\frac{\partial v}{\partial n}]_{|\Gamma^\varepsilon} = \frac{\partial v}{\partial n_e} + \frac{\partial v}{\partial n_i}$ with n_e and n_i the normal vector fields Γ^ε respectively outgoing and ingoing with respect to Ω^i_ε. Applying the formula of integral representation (see for instance [16]) we obtain

$$u^\varepsilon(\mathbf{x}) - u_0(\mathbf{x}) = \int_{\Gamma^\varepsilon} p^\varepsilon(\mathbf{x}')\, G(\mathbf{x},\mathbf{x}')\, d\sigma(\mathbf{x}') \quad \text{for any } \mathbf{x} \in \Omega_R.$$

Putting this formula into the equation corresponding to boundary condition satisfied by $u^\varepsilon - u_0$, we obtain

$$\int_{\Gamma^\varepsilon}\int_{\Gamma^\varepsilon} q(\mathbf{x})\, p^\varepsilon(\mathbf{x}')\, G(\mathbf{x},\mathbf{x}')\, d\sigma(\mathbf{x}')\, d\sigma(\mathbf{x}) = -\int_{\Gamma^\varepsilon} q(\mathbf{x})\, u_0(\mathbf{x})\, d\sigma(\mathbf{x}) \quad \forall q \in H^{-1/2}(\Gamma^\varepsilon).$$

We introduce the single layer operator $K^\varepsilon_{\rm sc} : H^{-1/2}(\Gamma^\varepsilon) \to H^{1/2}(\Gamma^\varepsilon)$ defined by

$$\forall\, p,q \in H^{-1/2}(\Gamma^\varepsilon), \quad \langle q\,,\, K^\varepsilon_{\rm sc}\, p\rangle_{H^{-1/2}(\Gamma^\varepsilon),H^{1/2}(\Gamma^\varepsilon)} = \int_{\Gamma^\varepsilon}\int_{\Gamma^\varepsilon} q(\mathbf{x})\, p(\mathbf{x}')\, G(\mathbf{x},\mathbf{x}')\, d\sigma(\mathbf{x}')\, d\sigma(\mathbf{x}).$$

We can write the above equation in operator form,

$$K^\varepsilon_{\rm sc}\, p^\varepsilon = -u_0.$$

Note that using Taylor expansion it is possible to prove that, if $\mathcal{O}$ is the open set defined in §2.3.3, then there exists $\kappa > 0$ independant of $\varepsilon > 0$ such that for any $v \in C^\infty(\mathcal{O})$, the following inequalities hold

$$\|v - \sigma^\varepsilon \circ \mu^\varepsilon(v)\|_{H^{1/2}(\Gamma^\varepsilon)} \leqslant \kappa\varepsilon\|v\|_{H^2(\mathcal{O})} \quad \text{et} \quad \|v_{|J} - \sigma^\varepsilon \circ \mu^\varepsilon(v)\|_{E^{1/2}(I)} \leqslant \kappa\varepsilon\|v\|_{H^2(\mathcal{O})}$$

In particular this remark can be applied to u_0. Thus we are led to

$$\begin{aligned} -u_0 = -\sigma^\varepsilon \circ \mu^\varepsilon(u_0) + O(\varepsilon) \quad &= \sigma^\varepsilon.\,(\ln \tfrac{1}{\varepsilon^2} Id + A + B).\,{}^t\sigma^\varepsilon(\widetilde{p}^\varepsilon) + O(\varepsilon) \\ &= \sigma^\varepsilon.\,(\ln \tfrac{1}{\varepsilon^2} Id + A + B).\,{}^t\sigma^\varepsilon(p^\varepsilon) + O(\varepsilon^{1/2}) \end{aligned}$$

Here the notation $O(\varepsilon)$ refers to the topology of $H^{1/2}(\Gamma^\varepsilon)$ defined in paragraph 2.4.3. So we obatin

$$K^\varepsilon_{\rm sc}\, p^\varepsilon = \sigma^\varepsilon.\,(\ln \frac{1}{\varepsilon^2} Id + A + B).\,{}^t\sigma^\varepsilon\, p^\varepsilon + O(\varepsilon^{1/2}).$$

8.3 New approximate model

We introduce a new space that corresponds to the set of functions with an average equal to 0 on each section of Γ^ε. Note that $\mathcal{H}^\varepsilon_\mu = \{v \in \mathcal{H} \mid \mu^\varepsilon(v_{|\Gamma^\varepsilon}) = 0\}$. It is a closed sub-space of $\mathcal{H}$ since μ^ε is continuous. Suppose that is supplied with the hermitian structure inherited from $\mathcal{H}$. Similarly to what precedes, we are led to consider the problem

$$(\mathbf{P}^\varepsilon) : \begin{cases} \text{Find} \quad (\mathfrak{u}^\varepsilon;\mathfrak{p}^\varepsilon) \in \mathcal{H} \times \mathcal{M}^{\varepsilon'} \quad \text{such that} \\ \displaystyle\int_{\Omega_R} \nabla\mathfrak{u}^\varepsilon \cdot \nabla\overline{v} - k^2 \int_{\Omega_R} \mathfrak{u}^\varepsilon\, \overline{v} + \int_{\Gamma_R} \overline{v}\, T_R \mathfrak{u}^\varepsilon + \int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(\mathfrak{p}^\varepsilon)\, \overline{v} = -\int_{\Omega_R} f\, \overline{v} \qquad \forall v \in \mathcal{H}, \\ \displaystyle\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(q)\, \overline{\mathfrak{u}^\varepsilon} = 0 \qquad \forall q \in \mathcal{M}^{\varepsilon'}. \end{cases}$$

We conjecture that $\mathfrak{u}^\varepsilon$ is a good approximation of u^ε. Indeed we state theorem 8.2 that we are going to prove with the same technics as for theorem 7.1.

Theorem 8.2
Let $(\mathfrak{u}^\varepsilon, \mathfrak{p}^\varepsilon) \in \mathcal{H} \times \mathcal{M}^{\varepsilon'}$ be the unique solution to problem $(\mathbf{P}^\varepsilon)$. Then there exists a constant $\kappa > 0$ independant of ε such that for ε small enough,

$$\|u^\varepsilon - \mathfrak{u}^\varepsilon\|_{\mathcal{H}} \leqslant \kappa\, \varepsilon^{1/2} \ln 1/\varepsilon \quad \textit{and} \quad \|p^\varepsilon - {}^t\mu^\varepsilon(\mathfrak{p}^\varepsilon)\|_{H^{-1/2}(\Gamma^\varepsilon)} \leqslant \kappa\, \varepsilon^{1/2} \ln 1/\varepsilon.$$

This theorem is interesting for two reasons. First it states that solving $(\mathcal{P}^\varepsilon)$ yields nearly the same result as solving $(\mathbf{P}^\varepsilon)$ whereas is problem $(\mathbf{P}^\varepsilon)$ the boundary condition on the wire is simplified, the space of Lagrange multipliers is reduced. This is interesting for numerical applications since the constraint corresponding to the boundary condition is now one dimensional instead of being two dimensional. More over we will see at the end of paragraph 8.4, that the error estimate $\|p^\varepsilon - {}^t\mu^\varepsilon(\mathfrak{p}^\varepsilon)\|_{H^{-1/2}(\Gamma^\varepsilon)}$ is a validation for the acoustic version of the Pocklington model. To the best of our knowledge, this result is new.

First of all we state a stability result . However since the proof is very similar to the proof of lemma 7.3, we do not write it. Given a fixed $\varepsilon > 0$ for $f^\varepsilon \in \mathcal{H}'$ and $g^\varepsilon \in \mathcal{M}^\varepsilon$, we consider the problem

$$\left\{\begin{array}{l} \text{Find} \quad (v^\varepsilon; q^\varepsilon) \in \mathcal{H} \times \mathcal{M}^{\varepsilon\,\prime} \quad \text{such that} \\ \displaystyle\int_{\Omega_R} \nabla v^\varepsilon \cdot \nabla \overline{v} - k^2 \int_{\Omega_R} v^\varepsilon\, \overline{v} + \int_{\Gamma_R} \overline{v}\, T_R v^\varepsilon + \int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(q^\varepsilon)\, \overline{v} = \langle f^\varepsilon; v\rangle_{\mathcal{H}',\mathcal{H}} \qquad \forall v \in \mathcal{H} \\ \displaystyle\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(q)\, \overline{v^\varepsilon} = \langle g^\varepsilon; v\rangle_{\mathcal{M}^{\varepsilon\prime},\mathcal{M}^\varepsilon} \qquad \forall q \in \mathcal{M}^{\varepsilon\,\prime} \end{array}\right. \tag{18}$$

The corresponding stability lemma is

Lemma 8.2
For ε small enough and for any $f^\varepsilon \in \mathcal{H}'$, $g^\varepsilon \in \mathcal{M}^\varepsilon$, the problem (18) admits a unique solution $(v^\varepsilon; q^\varepsilon) \in \mathcal{H} \times \mathcal{M}^{\varepsilon\,\prime}$ that satisfies in addition

$$\begin{aligned} \|v^\varepsilon\|_{\mathcal{H}} &\leqslant \frac{1}{\kappa_1}\|f^\varepsilon\|_{\mathcal{H}'} + (1 + \frac{\kappa_2}{\kappa_1})\|g^\varepsilon\|_{\mathcal{M}^\varepsilon}, \\ \|q^\varepsilon\|_{\mathcal{M}^{\varepsilon\prime}} &\leqslant (1 + \frac{\kappa_2}{\kappa_1})\|f^\varepsilon\|_{\mathcal{H}'} + \kappa_2(1 + \frac{\kappa_2}{\kappa_1})\|g^\varepsilon\|_{\mathcal{M}^\varepsilon}. \end{aligned}$$

A consequence of this lemma is that $(\mathbf{P}^\varepsilon)$ is well posed so $(\mathfrak{u}^\varepsilon, \mathfrak{p}^\varepsilon)$ is well defined.

8.4 Consistancy of the new wire model

We will now prove theorem 8.2. We have already pointed out that $H^{1/2}_\mu(\Gamma^\varepsilon) \subset Ker\,\widetilde{p}^\varepsilon$. So $\widetilde{p}^\varepsilon$ induces a linear form on $\mathcal{M}^\varepsilon$ that is different from a^ε. Indeed,

$$\begin{aligned} \langle \widetilde{p}^\varepsilon, v\rangle_{H^{-1/2}(\Gamma^\varepsilon),H^{1/2}(\Gamma^\varepsilon)} &= -\frac{\alpha_\varepsilon}{2\pi}\left\langle {}^t\mu^\varepsilon(a^\varepsilon), v\right\rangle_{H^{-1/2}(\Gamma^\varepsilon),H^{1/2}(\Gamma^\varepsilon)} \\ &= -\frac{\alpha_\varepsilon}{2\pi}\langle a^\varepsilon, \mu^\varepsilon(v)\rangle_{H^{-1/2}(\Gamma^\varepsilon),H^{1/2}(\Gamma^\varepsilon)}\,, \quad \forall v \in H^{1/2}(\Gamma^\varepsilon). \end{aligned}$$

Thus we can say that $(\widetilde{u}^\varepsilon, a^\varepsilon)$ satisfies

$$\left\{\begin{array}{l} (\widetilde{u}^\varepsilon; -\dfrac{\alpha_\varepsilon}{2\pi}\, a^\varepsilon) \in \mathcal{H} \times \mathcal{M}^{\varepsilon\,\prime} \quad \text{such that} \\ \displaystyle\int_{\Omega_R} \nabla \widetilde{u}^\varepsilon \cdot \nabla \overline{v} - k^2 \int_{\Omega_R} \widetilde{u}^\varepsilon\, \overline{v} + \int_{\Gamma_R} \overline{v}\, T_R \widetilde{u}^\varepsilon + \int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(-\frac{\alpha_\varepsilon}{2\pi}\, a^\varepsilon)\, \overline{v} = -\int_{\Omega_R} f\, \overline{v} + \left\langle \widetilde{f}^\varepsilon, v\right\rangle_{\mathcal{H}',\mathcal{H}} \qquad \forall v \in \mathcal{H} \\ \displaystyle\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(q)\, \overline{\widetilde{u}^\varepsilon} = 0 \qquad \forall q \in \mathcal{M}^{\varepsilon\,\prime}. \end{array}\right.$$

Note that $\widetilde{u}^\varepsilon_\delta = \widetilde{u}^\varepsilon - \mathfrak{u}^\varepsilon$ and $\widetilde{p}^\varepsilon_\delta = -\dfrac{\alpha_\varepsilon}{2\pi}\, a^\varepsilon - \mathfrak{p}^\varepsilon$. This new pair of functions satisfies

$$\left\{\begin{array}{l} (\widetilde{u}^\varepsilon_\delta; \widetilde{p}^\varepsilon_\delta) \in \mathcal{H} \times \mathcal{M}^{\varepsilon\,\prime} \quad \text{such that} \\ \displaystyle\int_{\Omega_R} \nabla \widetilde{u}^\varepsilon_\delta \cdot \nabla \overline{v} - k^2 \int_{\Omega_R} \widetilde{u}^\varepsilon_\delta\, \overline{v} + \int_{\Gamma_R} \overline{v}\, T_R \widetilde{u}^\varepsilon_\delta + \int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(\widetilde{p}^\varepsilon_\delta)\, \overline{v} = \left\langle \widetilde{f}^\varepsilon, v\right\rangle_{\mathcal{H}',\mathcal{H}} \qquad \forall v \in \mathcal{H}, \\ \displaystyle\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(q)\, \overline{\widetilde{u}^\varepsilon_\delta} = 0 \qquad \forall q \in \mathcal{M}^{\varepsilon\,\prime}. \end{array}\right.$$

Applying lemma 8.2 to the above problem, we obtain the existence of $\kappa > 0$ independant of ε such that for ε small enough

$$\|\widetilde{u}^\varepsilon - \mathfrak{u}^\varepsilon\|_{\mathcal{H}} \leqslant \kappa \|\widetilde{f}^\varepsilon\|_{\mathcal{H}'} \quad \text{et} \quad \|-\frac{\alpha_\varepsilon}{2\pi} a^\varepsilon - \mathfrak{p}^\varepsilon\|_{\mathcal{M}^{\varepsilon'}} \leqslant \kappa \|\widetilde{f}^\varepsilon\|_{\mathcal{H}'}. \tag{19}$$

Using the proof of theorem 7.1, we can prove the existence of κ independant of ε such that for ε small enough $\|\widetilde{f}^\varepsilon\|_{\mathcal{H}'} \leqslant \kappa\, \varepsilon^{1/2} \ln 1/\varepsilon$. With triangular inequality and lemma 8.1, this yields the proof of theorem 8.2. We will apply this result to the Pocklington model. Let us first recall that $(\mathfrak{u}^\varepsilon, \mathfrak{p}^\varepsilon) \in H^1(\Omega_R) \times \mathcal{M}^{\varepsilon'}$ is defined as the unique solution to the problem

$$(\mathbf{P}^\varepsilon) : \begin{cases} \displaystyle\int_{\Omega_R} \nabla \mathfrak{u}^\varepsilon \cdot \nabla \overline{v} - k^2 \int_{\Omega_R} \mathfrak{u}^\varepsilon \overline{v} + \int_{\Gamma_R} \overline{v} T_R \mathfrak{u}^\varepsilon + \int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(\mathfrak{p}^\varepsilon)\overline{v} = -\int_{\Omega_R} f\overline{v} \quad \forall v \in \mathcal{H}, \\ \displaystyle\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(q)\mathfrak{u}^\varepsilon = 0 \quad \forall q \in \mathcal{F}^{\varepsilon'}. \end{cases}$$

Let us substract to these equations the equations satisfied by u_0, this gives

$$\begin{cases} \displaystyle\int_{\Omega_R} \nabla(\mathfrak{u}^\varepsilon - u_0) \cdot \nabla \overline{v} - k^2 \int_{\Omega_R} (\mathfrak{u}^\varepsilon - u_0)\overline{v} + \int_{\Gamma_R} \overline{v} T_R(\mathfrak{u}^\varepsilon - u_0) + \int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(\mathfrak{p}^\varepsilon)\overline{v} = 0 \quad \forall v \in \mathcal{H}, \\ \displaystyle\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(q)(\mathfrak{u}^\varepsilon - u_0) = -\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(q) u_0 \quad \forall q \in H^{-1/2}(\Gamma^\varepsilon). \end{cases}$$

Successively choosing $v \in \mathcal{D}(\Omega_\varepsilon^i)$ and $v \in \mathcal{D}(\Omega_\varepsilon^R)$ arbitrarily, this implies that $\Delta(\mathfrak{u}^\varepsilon - u_0) + k^2(\mathfrak{u}^\varepsilon - u_0) = 0$ in Ω_ε^i and in Ω_ε^R, and $\mathfrak{u}^\varepsilon - u_0$ outgoing. Choosing $v \in \mathcal{D}(\Omega_R)$ arbitrarily and appplying Green's formula, we obtain

$$[\frac{\partial(\mathfrak{u}^\varepsilon - u_0)}{\partial n}]_{|\Gamma^\varepsilon} = {}^t\mu^\varepsilon(\mathfrak{p}^\varepsilon)$$

where $[\frac{\partial v}{\partial n}]_{|\Gamma^\varepsilon} = \frac{\partial v}{\partial n_e} + \frac{\partial v}{\partial n_i}$, n_e being n_i respectively the outgoing and ingoing normal vectors to Γ^ε. Applying integral representation formula yields

$$\mathfrak{u}^\varepsilon(\mathbf{x}) - u_0(\mathbf{x}) = -\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(\mathfrak{p}^\varepsilon)(\mathbf{x}')\, G(\mathbf{x}, \mathbf{x}')\, d\sigma(\mathbf{x}') \quad \text{pour tout } \mathbf{x} \in \Omega_R.$$

Putting this formula into the equation corresponding to the boundary condition satisfied by $\mathfrak{u}^\varepsilon$, we obtain

$$\int_{\Gamma^\varepsilon}\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(q)(\mathbf{x})\; {}^t\mu^\varepsilon(\mathfrak{p}^\varepsilon)(\mathbf{x}')\, G(\mathbf{x}, \mathbf{x}')\, d\sigma(\mathbf{x}')\, d\sigma(\mathbf{x}) = \int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(q)(\mathbf{x})\, u_0(\mathbf{x})\, d\sigma(\mathbf{x}) \quad \forall q \in H^{-1/2}(\Gamma^\varepsilon)$$

Define the operator $K^\varepsilon_{\text{pk}} : E^{-1/2}(I) \to E^{1/2}(I)$ by

$$\forall p, q \in E^{-1/2}(I), \quad \left\langle q\,,\, K^\varepsilon_{\text{pk}}\, p\right\rangle_{E^{-1/2}(I), E^{1/2}(I)} = \int_{\Gamma^\varepsilon}\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(q)(\mathbf{x})\; {}^t\mu^\varepsilon(p)(\mathbf{x}')\, G(\mathbf{x}, \mathbf{x}')\, d\sigma(\mathbf{x}')\, d\sigma(\mathbf{x})$$

Thus we can write the above equation in operator form

$$K^\varepsilon_{\text{pk}}\, \mathfrak{p}^\varepsilon = \mu^\varepsilon(u_0)$$

$$\begin{aligned} \mu^\varepsilon(u_0) &= (\ln \tfrac{1}{\varepsilon^2} Id + A + B).\, {}^t\sigma^\varepsilon(\widetilde{p}^\varepsilon) + O(\varepsilon) \\ &= (\ln \tfrac{1}{\varepsilon^2} Id + A + B).\mathfrak{p}^\varepsilon + O(\varepsilon^{1/2}). \end{aligned}$$

This implies

$$K^\varepsilon_{\text{pk}}\, \mathfrak{p}^\varepsilon = (\ln \frac{1}{\varepsilon^2} Id + A + B)\, \mathfrak{p}^\varepsilon + O(\varepsilon^{1/2}).$$

rem: Note that according to the preceding equations, there exists an explicit link between the usual single layer operator $K^\varepsilon_{\text{sc}}$ and the Pocklington operator $K^\varepsilon_{\text{pk}}$,

$$K^\varepsilon_{\text{pk}} = \mu^\varepsilon \circ K^\varepsilon_{\text{sc}} \circ {}^t\mu^\varepsilon.$$

A Appendix

A.1 Dirichlet-to-Neumann map

In this paragraph we recall the definition and some properties of the Dirichlet-to-Neumann map used in practice for the treatment of Helmholtz equation in the exterior of a bounded obstacle in three dimensions. For more details on this subject we refer the reader to [15]. Consider a radius $R > 0$ and the associated sphere ∂B_R. On this surface we consider the spherical coordinates (ν, θ). Then according to classical results on spherical harmonics,

$$u \in H^{1/2}(\partial B_R) \quad \text{if and only if there exists } (u_l^m) \text{ such that}$$

$$\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}(1+l^2)^{1/2}|u_l^m|^2 < +\infty \quad \text{and} \quad u(\theta;\varphi) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l} u_l^m\,\widetilde{P}_l^m(\nu)e^{im\theta}$$

the convergence of the second sum above holds in $L^2(\partial B_R)$. Then the usual Dirichlet-to-Neumann map is given by the explicit formula,

$$(T_R u)(\nu,\theta) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l} u_l^m\,P_l^m(\nu)e^{im\theta}\,k\,\frac{h_m^{(1)'}(kR)}{h_m^{(1)}(kR)}.$$

This formula defines T_R as a continuous map from $H^{1/2}(\partial B_R)$ onto $H^{-1/2}(\partial B_R)$. Moreover we have the remarkable inequality,

$$\Re e\{\int_{\partial B_R}\overline{v}T_R v\} \geqslant 0\,, \quad \forall v \in H^{1/2}(\partial B_R).$$

A.2 Legendre functions

Here we formulate several remarks on the separation of variables for the Laplace equation (an not for the Helmholtz equation) in ellipsoidal coordinates. Indeed the asymptotic analysis for a differential operator in a simple geometry leads to consider special functions associated to this operator and to the coordinate system adapted to the geometry. The natural approach consists then in studying the expansions of these special functions, possible representations (integral representation, series representation, and so on ...). As a consequence, the better these special functions are known, the more we know about the asymptotic analysis. In our case, the Helmholtz equation can be solved in ellipsoidal coordinates using ellipsoidal wave functions. The problem is that these functions are not so well known. We prefered to look at the Helmholtz operator as the Laplace operator with a compact perturbation. The Laplace operator can also be solved by separation of variables and the corresponding special functions are the Legendre functions for which we have far more information. Legendre functions are very well known. Given an integer $n \in \mathbb{N}$ consider the Legendre equation of order n:

$$\frac{d}{dz}(1-z^2)\frac{d}{dz}u(z) + n(n+1)u(z) = 0$$

In [19],chap.7 it is proved that this equation admits two independant solutions that we denote, on the one hand P_n the Legendre function of order n of the first kind, and the other hand Q_n the Legendre functions of order n of the second kind. It can be proved that P_n is a polynomial of order n called Legendre polynomial nd given explicitly by the Rodrigues formula

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n$$

In [19], it is also proved that

$$Q_n(z) = \frac{P_n(z)}{2}\ln\Big(\frac{z+1}{z-1}\Big) + W_{n-1}(z)$$

where W_{n-1} is a polynomial of order $n-1$.. Robin, in [17] wrote a very simple identity for W_{n-1}, expressed with Legendre polynomials

$$W_{n-1}(z) = -\sum_{k=0}^{n-1} \frac{P_k(z)P_{n-1-k}(z)}{k+1} \tag{20}$$

Now we recall some further properties and estimates on P_n. The proof of these results can be found in [19] or in [11]. In what follows we denote $I = [-1;+1]$. Consider $L^2(I)$ supplied with its usual scalar product $\langle u; v\rangle_{L^2(I)} = \int_I u\,\overline{v}$. It can be proved that $(P_n)_{n\in\mathbb{N}}$ is an orthonormal family, and the set of linear combinations of these polynomials is dense in $L^2(I)$. The L^2 norm of P_n on I is given by

$$\int_{-1}^{+1} |P_n(x)|^2\,dx = \frac{1}{n+1/2}$$

In what follows we consider the normalized Legendre polynomials $\widetilde{P}_n = \sqrt{n+1/2}\,P_n$. They satisfy the identity

$$\begin{aligned} xP_n(x) &= \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x) && n \geqslant 1 \\ (1-x^2)\frac{dP_n}{dx} &= -\frac{n(n+1)}{2n+1}P_{n+1}(x) + \frac{n(n+1)}{2n+1}P_{n-1}(x) && n \geqslant 1. \end{aligned}$$

The value of the Legendre polynomial at 1 is $P_n(1) = 1\ \forall n \in \mathbb{N}$. Moreover we have the estimate

$$|P_n(x)| \leqslant 1 \quad \text{et} \quad |\frac{dP_n}{dx}(x)| \leqslant \frac{n(n+1)}{2} \qquad \forall x \in [-1;+1].$$

Now we show two results. The first one is fundamental for our analysis

Lemma A.1

$$\int_{-1}^{+1} \frac{P_n(z)\,dz}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}} = 2P_n(\nu)Q_n(\xi)$$

Proof:

For $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$ define $G(\mathbf{x}) = \dfrac{1}{4\pi|\mathbf{x}|}$. G is a Green kernel of the Laplace equation in $\mathbb{R}^3$. Thus if $\mathbf{x}'$ has the cartesian coordinates $(0;0;z)$ and $\mathbf{x}$ has the ellipsoidal coordinates $(\xi;\nu;\theta)$ then

$$G(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} = \frac{1}{4\pi\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}}$$

As a consequence if u is defined by

$$u(\xi;\nu) = \int_{-1}^{+1} \frac{P_n(z)\,dz}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}} \quad , \quad \xi \in]1;+\infty[\quad \nu \in [-1;+1]$$

then u satisfies the Laplace equation in ellipsoidal coordinates, and since u is independant of θ,

$$\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial u}{\partial\xi} + \frac{\partial}{\partial\nu}(1-\nu^2)\frac{\partial u}{\partial\nu} = 0 \quad , \quad \xi \in]1;+\infty[\quad \nu \in [-1;+1].$$

Let us denote $u_l(\xi) = \int_I u(\xi;\nu)\widetilde{P}_l(\nu)\,d\nu$. Taking the projection of this equation on P_l, we obtain

$$\int_I \frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial u}{\partial\xi}\,\widetilde{P}_l(\nu)\,d\nu + \int_I \frac{\partial}{\partial\nu}(1-\nu^2)\frac{\partial u}{\partial\nu}\,\widetilde{P}_l(\nu)\,d\nu = 0 \quad , \quad \xi \in]1;+\infty[.$$

For fixed ξ, since u is a C^∞ function it is possible to switch the integral and the differential operator in the first integral. Moreover $\widetilde{P}_l$ satisfies the Legendre equation associated with l so integrating by parts in the second integral we obtain

$$\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial u_l}{\partial\xi}+l(l+1)u_l(\xi)=0$$

As a consequence u_l satisfies a Legendre associated to l on $]1;+\infty[$ so there exists $A_l, B_l \in \mathbb{C}$ such that $u_l(\xi) = A_l P_l(\xi) + B_l Q_l(\xi)$. We can rewrite this identity int he following manner

$$A_l P_l(\xi) = \int_I u(\xi;\nu)\widetilde{P}_l(\nu)\,d\nu - B_l Q_l(\xi)$$

Taking into account the estimate we gave previously for P_n, we have

$$\Big|\int_I u(\xi;\nu)\widetilde{P}_l(\nu)\,d\nu\Big| = \Big|\int_I\int_I \frac{P_n(z)\widetilde{P}_l(\nu)\,dzd\nu}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}}\Big| \leqslant \sqrt{\frac{l+1/2}{\xi^2-1}}\int_I\frac{d\nu}{\sqrt{1-\nu^2}} = \pi\sqrt{\frac{l+1/2}{\xi^2-1}} \underset{\xi\to+\infty}{\longrightarrow} 0$$

Moreover according to the following integral representation ([19] formula 7.4.9):

$$Q_l(\cosh\alpha) = \int_0^{+\infty}\frac{dx}{(\cosh\alpha+\sinh\alpha\cosh\alpha)^{l+1}} \leqslant \frac{1}{(\sinh\alpha)^{l+1}}\int_0^{+\infty}\frac{dx}{(\cosh x)^{l+1}}$$

hence there exists $C>0$ such that $\forall\xi\in[2;+\infty]$, $\forall l\in\mathbb{N}$:

$$|Q_l(\xi)| < \frac{C}{(\xi^2-1)^{\frac{l+1}{2}}} \underset{\xi\to+\infty}{\longrightarrow} 0.$$

To sum up $u_l(\xi) = A_l P_l(\xi) + B_l Q_l(\xi)$ with $|u_l(\xi)|\underset{\xi\to+\infty}{\longrightarrow} 0$, $|Q_l(\xi)|\underset{\xi\to+\infty}{\longrightarrow} 0$ and $|P_l(\xi)|\underset{\xi\to+\infty}{\longrightarrow} +\infty$.

To sum up: $\quad A_l = 0$

We will now compute B_l. For this purpose we look for the behavior of u_l when $\xi\to 1_+$.

$$\begin{aligned} u_l(\xi) &= \int_{I\times I}\frac{P_n(z)\widetilde{P}_l(\nu)\,dzd\nu}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}} \\ &= \int_I P_n(\nu)\widetilde{P}_l(\nu)\int_I\frac{dz}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}}d\nu + \int_I\widetilde{P}_l(\nu)\int_I\frac{P_n(z)-P_n(\nu)}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}}dzd\nu. \end{aligned}$$

Here is an explicit computation for the first integral

$$\begin{aligned} \int_{-1}^{+1}\frac{dz}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}} &= \int_{(-1-\xi\nu)/\sqrt{(\xi^2-1)(1-\nu^2)}}^{(1-\xi\nu)/\sqrt{(\xi^2-1)(1-\nu^2)}}\frac{dz}{\sqrt{1+z^2}} \\ &= \ln\Big(\frac{1-\xi\nu+\sqrt{(\xi^2-1)(1-\nu^2)+(1-\xi\nu)^2}}{-1-\xi\nu+\sqrt{(\xi^2-1)(1-\nu^2)+(-1-\xi\nu)^2}}\Big) \end{aligned}$$

Considering $\xi = 1+u$, let us simplify

$$\begin{aligned} (\xi^2-1)(1-\nu^2)+(1-\xi\nu)^2 &= u(2+u)(1-\nu^2)+(1-\nu-\nu u)^2 \\ &= u^2(1-\nu^2)+2u(1-\nu^2)+\nu^2u^2-2u\nu(1-\nu)+(1-\nu)^2 \\ &= u^2+2u(1-\nu)+(1-\nu)^2 = (1+u-\nu)^2 = (\xi-\nu)^2. \end{aligned}$$

With the same computation we obtain $(\xi^2-1)(1-\nu^2)+(1+\xi\nu)^2 = (\xi+\nu)^2$. Putting these identities in the logarithmic term above,

$$\int_{-1}^{+1}\frac{dz}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}} = \ln\Big(\frac{1-\xi\nu+\xi-\nu}{-1-\xi\nu+\xi+\nu}\Big) = \ln\Big(\frac{\xi+1}{\xi-1}\Big) \tag{21}$$

To sum up: $\int_I P_n(\nu)\widetilde{P}_l(\nu)\int_I \frac{dz}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}}d\nu = \frac{\delta_n^l}{\sqrt{n+1/2}}\ln\Big(\frac{\xi+1}{\xi-1}\Big).$

Besides, for given ν and z, we have

$$\frac{P_n(z)-P_n(\nu)}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}} \underset{\xi\to 1_+}{\longrightarrow} \frac{P_n(z)-P_n(\nu)}{|z-\nu|}.$$

Since $(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2 = (\xi-1)\underbrace{(\xi-\nu z+1-\nu z)}_{\geqslant 0}+(\nu-z)^2 \geqslant (\nu-z)^2$ il vient:

$$\frac{|P_n(z)-P_n(\nu)|}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}} \leqslant \Big|\frac{P_n(z)-P_n(\nu)}{z-\nu}\Big| \leqslant \max_{\bar I}|P_n'|$$

The dominated convergence theorem yields

$$\int_I \widetilde{P}_l(\nu)\int_I \frac{P_n(z)-P_n(\nu)}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}}dzd\nu \underset{\xi\to 1_+}{\longrightarrow} \int_I \widetilde{P}_l(\nu)\int_I \frac{P_n(z)-P_n(\nu)}{|z-\nu|}\,dzd\nu.$$

Finally we have obtained

$$\begin{aligned} u_l(\xi) &= \frac{\delta_n^l}{\sqrt{n+1/2}}\ln\Big(\frac{\xi+1}{\xi-1}\Big)+\int_I \widetilde{P}_l(\nu)\int_I \frac{P_n(z)-P_n(\nu)}{|z-\nu|}\,dzd\nu + o(1) \\ &= B_l\Big(P_l(\xi)\frac{1}{2}\ln(\frac{\xi+1}{\xi-1})+W_{l-1}(\xi)\Big). \end{aligned}$$

P_l and W_{l-1} are polynomials and $P_l(1)=1$. Thus identifying the respective behaviors in the neighborhood of 1_+ we obtain

Conclusion: $\qquad B_l = \frac{2\delta_n^l}{\sqrt{n+1/2}}.$

Now we conclude the proof of the lemma. For $\xi > 1$ fixed, and $\nu \mapsto u(\xi;\nu)$ is continuous on $[-1;+1]$, theorem 1 p.55 of [19] provides

$$u(\xi;\nu) = \sum_{l=0}^{+\infty} u_l(\xi)\widetilde{P}_l(\nu) = \sum_{l=0}^{+\infty} B_l\widetilde{P}_l(\nu)Q_l(\xi) = \frac{2}{\sqrt{n+1/2}}\widetilde{P}_n(\nu)Q_n(\xi) = 2P_n(\nu)Q_n(\xi).$$

■

Corollary A.1
For any $\xi > 1$, $\nu \in [-1;+1]$, $z \in [-1;+1]$*:*

$$\frac{1}{\sqrt{(\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2}} = \sum_{n=0}^{+\infty} 2\widetilde{P}_n(z)\widetilde{P}_n(\nu)Q_n(\xi)$$

convergence being pointwise.

Proof:
For $\xi > 1$, $\nu \in [-1;+1]$ fixed, the function $z \mapsto ((\xi^2-1)(1-\nu^2)+(\xi\nu-z)^2)^{-1/2}$ is continuous on $[-1;+1]$. We conclude the proof applying lemma A.1 and theorem Il suffit à présent d'appliquer le lemme 1 p.55 of [19].

■

Using lemma A.1, we can describe precisely the Green kernel of the Laplace equation in ellipsoidal coordinates. We end this section with technical lemmas providing estimates on Q_n. In what follows $[x]$ refers to the greater integer satisfying $[x] \leqslant x$.

Lemma A.2
For $\varepsilon_0 > 0$ smal enough, there exists $C > 0$ independant of $\varepsilon \in]0;\varepsilon_0[$, $\xi \in [1;+\infty[$ ans $n \in \mathbb{N}$ such that

$$|P_n(\xi) - 1| \leqslant C\, n\sqrt{\xi^2 - 1} \quad \textit{dès que} \quad n \leqslant [\frac{1}{\sqrt{\varepsilon}}] \quad \textit{et} \quad \varepsilon \leqslant \xi^2 - 1 \leqslant 2\varepsilon$$

Proof:
We will use a result given in [20] chap.12 §12.3. This result states that

$$P_n(\cosh x) = \Big(\frac{x}{\sinh x}\Big)^{1/2}\Big(I_0(x(n+\frac{1}{2})) + \eta_{1,1}(n+\frac{1}{2};x^2)\Big)$$

with

$$|\eta_{1,1}(n+\frac{1}{2};x^2)| \leqslant I_0(x(n+\frac{1}{2}))\ \frac{\lambda_1(0)}{n+\frac{1}{2}}\mathcal{V}_{0,x}(\tau B_0^0(\tau^2))\exp\Big(\frac{\lambda_1(0)}{n+\frac{1}{2}}\mathcal{V}_{0,x}(\tau B_0^0(\tau^2))\Big)$$

$$\text{where}\quad \begin{cases} I_0(x) = J_0(ix) \quad \text{and } J_0 \text{ refers to the Bessel function of the first kind of order } 0 \\ \lambda_1(0) = \sup\limits_{x\in]0;+\infty[} |2xI_0(x)K_0(x)| \\ \mathcal{V}_{0,x}(f) \quad \text{refers to the total variation of } f \text{ on }]0;x[\\ B_0^0(x) = \frac{-1}{16\sqrt{x}}\int_0^x \frac{1}{\sqrt{v}}\{\frac{1}{\sinh^2(\sqrt{v})} - \frac{1}{v}\}\, dv \end{cases}$$

Since $\mathcal{V}_{0,x}(f) = \int_0^x |f'|$ when $f' \in L^1(]0;x[)$ and $\frac{d}{dx}(xB_0^0(x)) = \frac{1}{8}(\frac{1}{x^2} - \frac{1}{\sinh^2(x)}) = \widetilde{B}(x)$, Taking $C_1 = \max\limits_{[0;1]} |\widetilde{B}|$ we obtain

$$\mathcal{V}_{0,x}(\tau B_0^0(\tau^2)) \leqslant x\int_0^1 |\widetilde{B}(tx)|dt \leqslant C_1 x$$

Thus we are led to the existence of $C_2 > 0$ independant of x and n such that

$$|\eta_{1,1}(n+\frac{1}{2};x^2)| \leqslant \frac{C_2}{n+\frac{1}{2}} I_0(x(n+\frac{1}{2}))x.$$

As a consequence

$$P_n(\cosh x) - 1 = \Big[\Big(\frac{x}{\sinh x}\Big)^{1/2} - 1\Big]\, I_0(x(n+\frac{1}{2})) + I_0(x(n+\frac{1}{2})) - 1 + \Big(\frac{x}{\sinh x}\Big)^{1/2}\eta_{1,1}(n+\frac{1}{2};x^2).$$

Since the function $x \mapsto (\frac{x}{\sinh x})^{1/2}$ is analytic in the neighborhood of 0 and take the value 1 in 0, there exists $C_3 > 0$ such that $|(\frac{x}{\sinh x})^{1/2} - 1| \leqslant C_3 x$. For $\varepsilon \leqslant \xi^2 - 1 \leqslant 2\varepsilon$ and $x = arcosh(\xi)$, and $n \leqslant [\frac{1}{\sqrt{\varepsilon}}]$ we have

$$(n+\frac{1}{2})x \leqslant \frac{1}{\sqrt{\varepsilon}}\ln(1+\xi-1+\sqrt{\xi^2-1}) \leqslant \frac{2\sqrt{2}\sqrt{\xi-1}}{\sqrt{\varepsilon}} \leqslant 4.$$

Take $C_4 = \max\limits_{[0;4]} |I_0| + |I_0'|$. Then we obtain the estimates

$$\begin{aligned} &|\Big[\Big(\frac{x}{\sinh x}\Big)^{1/2} - 1\Big]\, I_0(x(n+\frac{1}{2}))| \leqslant C_3C_4\, x, \\ &|I_0(x(n+\frac{1}{2})) - 1| \leqslant C_4(n+\frac{1}{2})\, x, \\ &|\Big(\frac{x}{\sinh x}\Big)^{1/2}\eta_{1,1}(n+\frac{1}{2};x^2)| \leqslant C_2C_4\, \frac{x}{n+\frac{1}{2}}. \end{aligned}$$

To sum up there exists $C_5 > 0$ independant of x and n such that

$$|P_n(\cosh x) - 1| \leqslant C_5\,(n+\frac{1}{2})x \quad \text{for} \quad n \leqslant [\frac{1}{\sqrt{\varepsilon}}] \quad \text{and} \quad \varepsilon \leqslant (\cosh x)^2 - 1 \leqslant 2\varepsilon.$$

Since for $\xi = \cosh x$ there holds $x \leqslant 2\sqrt{\xi^2-1}$, there exists $C_6 > 0$ independant of ε, ξ and n such that

$$|P_n(\xi) - 1| \leqslant C_6\, n\sqrt{\xi^2-1} \quad \text{when} \quad n \leqslant [\frac{1}{\sqrt{\varepsilon}}] \quad \text{and} \quad \varepsilon \leqslant \xi^2 - 1 \leqslant 2\varepsilon.$$

■

Corollary A.2
For $\varepsilon_0 > 0$ small enough, there exists $\kappa > 0$ independant of $\varepsilon \in]0;\varepsilon_0[$, $\xi \in [1;+\infty[$ and $n \in \mathbb{N}$ such that

$$|\frac{\partial P_n}{\partial \xi}(\xi)| \leqslant \kappa\,\frac{n}{\sqrt{\xi^2-1}} \quad \textit{when} \quad n \leqslant [\frac{1}{\sqrt{\varepsilon}}] \quad \textit{and} \quad \varepsilon \leqslant \xi^2 - 1 \leqslant 2\varepsilon.$$

Proof:
Note that

$$\begin{aligned}\frac{\partial P_n}{\partial \xi} &= \frac{1}{\xi^2-1}\{nP_{n-1}(\xi) - n\xi P_n(\xi)\}\\ &= \frac{n}{\xi^2-1}(P_{n-1}(\xi)-1) - \frac{n}{\xi^2-1}(\xi-1)(P_n(\xi)-1) - \frac{n}{\xi^2-1}(\xi-1) - \frac{n}{\xi^2-1}(P_n(\xi)-1).\end{aligned}$$

We conclude the proof of this lemma using lemma A.2.

■

Corollary A.3
For $\varepsilon_0 > 0$ small enough, there exists $\kappa > 0$ independant of $\varepsilon \in]0;\varepsilon_0[$, $\xi \in [1;+\infty[$ and $n \in \mathbb{N}$ such that

$$|W_{n-1}(\xi) - W_{n-1}(1)| \leqslant \kappa\, n\, \ln n\,\sqrt{\xi^2-1} \quad \textit{when} \quad n \leqslant [\frac{1}{\sqrt{\varepsilon}}] \quad \textit{and} \quad \varepsilon \leqslant \xi^2 - 1 \leqslant 2\varepsilon.$$

Proof:
We begin with an identity we gave before

$$W_{n-1}(z) = \sum_{q=0}^{n-1} \frac{P_q(z)P_{n-1-q}(z)}{q+1}.$$

As a consequence

$$\begin{aligned}W_{n-1}(\xi) - W_{n-1}(1) = &\quad \textstyle\sum_{q=0}^{n-1} \frac{1}{q+1}\{P_q(\xi)-1\}\{P_{n-1-q}(\xi)-1\}\\ &+ \textstyle\sum_{q=0}^{n-1} \frac{1}{q+1}\{P_q(\xi)-1\} + \sum_{q=0}^{n-1} \frac{1}{q+1}\{P_{n-1-q}(\xi)-1\}.\end{aligned}$$

from which we conclude using the preceding lemma and the expansion

$$\sum_{q=0}^{n-1} \frac{1}{q+1} = \ln q + \gamma + o(1)$$

where γ is the Euler constant.

■

Corollary A.4
For $\varepsilon_0 > 0$ small enough, there exists $\kappa > 0$ independant of $\varepsilon \in]0;\varepsilon_0[$, $\xi \in [1;+\infty[$ and $n \in \mathbb{N}$ such that

$$|\frac{\partial W_{n-1}}{\partial \xi}(\xi)| \leqslant \kappa \frac{n \ln n}{\sqrt{\xi^2-1}} \quad \text{when} \quad n \leqslant [\frac{1}{\sqrt{\varepsilon}}] \quad \text{and} \quad \varepsilon \leqslant \xi^2 - 1 \leqslant 2\varepsilon$$

Proof:
Straightforwardly

$$\frac{\partial W_{n-1}}{\partial \xi} = \sum_{l=0}^{n-1} \frac{1}{l+1}\{\frac{\partial P_l}{\partial \xi} P_{n-1-l} + \frac{\partial P_{n-1-l}}{\partial \xi} P_l\}.$$

Since there exists $\kappa > 0$ such that

$$|P_l(\xi)| \leqslant \kappa \quad \text{and} \quad |\frac{\partial P_l}{\partial \xi}(\xi)| \leqslant \kappa \frac{n}{\sqrt{\xi^2-1}} \quad \text{when} \quad l \leqslant n \leqslant [\frac{1}{\sqrt{\varepsilon}}] \quad \text{and} \quad \varepsilon \leqslant \xi^2 - 1 \leqslant 2\varepsilon.$$

From this we conclude

$$|\frac{\partial W_{n-1}}{\partial \xi}| \leqslant \kappa \frac{n}{\sqrt{\xi^2-1}} \sum_{l=0}^{n-1} \frac{1}{l+1} \leqslant \kappa \frac{n \ln n}{\sqrt{\xi^2-1}} \quad \text{when} \quad l \leqslant n \leqslant [\frac{1}{\sqrt{\varepsilon}}] \quad \text{and} \quad \varepsilon \leqslant \xi^2 - 1 \leqslant 2\varepsilon.$$

■

Using simple computations we also have

Corollary A.5
For $\varepsilon_0 > 0$ small enough, there exists $\kappa > 0$ independant of $\varepsilon \in]0;\varepsilon_0[$, $\xi \in [1;+\infty[$ and $n \in \mathbb{N}$ such that

$$|Q_n(\xi) - \frac{1}{2}\ln\Big(\frac{2}{\xi-1}\Big) - W_{n-1}(1)| \leqslant \kappa\, n \ln n \sqrt{\xi^2-1}\, \ln \frac{1}{\xi - 1}$$

when $n \leqslant [\frac{1}{\sqrt{\varepsilon}}]$ and $\varepsilon \leqslant \xi^2 - 1 \leqslant 2\varepsilon$.

Corollary A.6
For $\varepsilon_0 > 0$ small enough, there exists $\kappa > 0$ independant of $\varepsilon \in]0;\varepsilon_0[$, $\xi \in [1;+\infty[$ and $n \in \mathbb{N}$ such that

$$|\frac{\partial Q_n}{\partial \xi}(\xi) - \frac{1}{2}\frac{\partial}{\partial \xi}\ln\Big(\frac{2}{\xi-1}\Big)| \leqslant \kappa \frac{n \ln n}{\sqrt{\xi^2-1}} \ln \frac{1}{\xi-1}$$

when $n \leqslant [\frac{1}{\sqrt{\varepsilon}}]$ and $\varepsilon \leqslant \xi^2 - 1 \leqslant 2\varepsilon$.

A.3 Continuity of B_1, B_2 and B_3

Using a straightforward computational method, we will show that for any n, B_1 and B_2 are continuous $E^n(I)$ into $E^n(I)$. We first give the proof for B_1, and then for B_2 the proof will be very similar. We will proceed in two steps, using recurrence arguments. First we recall the defintion of B_1 B_2 and B_3,

$$\Big(B_1 u\Big)(\nu) = u(\nu) \ln c(\nu)^2,$$

$$\Big(B_2 u\Big)(\nu) = \int_{-1}^{+1} u(z) \frac{e^{ik|\nu - z|} - 1}{|\nu - z|} dz,$$

$$\Big(B_3 u\Big)(\nu) = \int_{-1}^{+1} u(z)\, G^{\mathrm{reg}}(\nu; z)\, dz.$$

A.3.1 Continuity of B_1

Lemma A.3
For $q = 0 \dots N$ suppose there exists $f_q, g_q \in C^\infty(\overline{I})$ and $u \in E^{2N}(I)$ such that

$$v = \sum_{q=0}^{N} f_q(\nu)\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^q u + \sum_{q=0}^{N-1} g_q(\nu)(1-\nu^2)\frac{d}{d\nu}\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^q u,$$

then there exists $F_q, G_q \in C^\infty(\overline{I}), q = 0 \dots N$ such that

$$\frac{d}{d\nu}(1-\nu^2)\frac{dv}{d\nu} = \sum_{q=0}^{N+1} F_s(\nu)\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^q u + \sum_{q=0}^{N} G_q(\nu)(1-\nu^2)\frac{d}{d\nu}\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^q u.$$

Proof:
We establish this result using recurrence on N. For $N = 0$, if $v = f_0(\nu)\, u$ then

$$\frac{d}{d\nu}(1-\nu^2)\frac{dv}{d\nu} = \frac{d}{d\nu}(1-\nu^2)\frac{df_0}{d\nu}\,.\,u + 2\frac{df_0}{d\nu}\,.\,(1-\nu^2)\frac{du}{d\nu} + f_0(\nu)\,.\,\frac{d}{d\nu}(1-\nu^2)\frac{du}{d\nu}$$

so the result is true for $N = 0$. Suppose now that the result is true for $q = 0 \dots N$. By assumption of recurrence, it is sufficient to prove that

$$(i) \qquad \frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big\{f_N(\nu)\,\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^N u\Big\}$$

$$(ii) \qquad \frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big\{g_N(\nu)\,(1-\nu^2)\frac{d}{d\nu}\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^{N-1} u\Big\}$$

have the expected form. This is a simple computational exercice. We write it for (i).

$$\begin{aligned}&\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big\{f_N(\nu)\,\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^N u\Big\} = \\ &\Big(\frac{d}{d\nu}(1-\nu^2)\frac{df_N}{d\nu}\Big)\,.\,\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^N u + 2\frac{df_N}{d\nu}\,.\,\Big\{(1-\nu^2)\frac{d}{d\nu}\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^N u\Big\} \\ &+ f_N(\nu)\,\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^{N+1} u\end{aligned}$$

which is the desired result.

■

Lemma A.4 $\forall n \in \mathbb{N}, \quad B_1 : E^n(I) \to E^n(I)$ *is continuous.*

Proof:
Consider $N \in \mathbb{N}$ and $n \in \{0 \dots N\}$. If n is even, $n = 2m$. According to the preceding lemma, there exists $f_q, g_q \in C^\infty(\overline{I}), q = 0 \dots m$ such that

$$\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^m (u(\nu)\ln c(\nu)) = \sum_{q=0}^{m} f_q(\nu)\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^q u + \sum_{q=0}^{m-1} g_q(\nu)(1-\nu^2)\frac{d}{d\nu}\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^q u$$

because $\ln c(\nu) = \ln 2 + v_0(\nu) \in C^\infty(\overline{I})$. Then

$$\begin{aligned}\Big|\big\langle(-\mathcal{L})^{2m} u(\nu)\ln c(\nu); u(\nu)\ln c(\nu)\big\rangle\Big|^{1/2} \quad &\leqslant \quad \sum_{q=0}^{m} \|f_q\|_{L^\infty(I)} |\big\langle(-\mathcal{L})^{2q}u; u\big\rangle|^{1/2} \\ &+ \sum_{q=0}^{m-1} \|(1-\nu^2)^{1/2} g_q\|_{L^\infty(I)} |\big\langle(-\mathcal{L})^{2q+1}u; u\big\rangle|^{1/2}.\end{aligned}$$

If n is odd, $n=2m+1$. There exists $f_q, g_q \in C^\infty(\overline{I}), q=0\dots m$ like in the preceding lemma. Then

$$\begin{aligned}\big\langle(-\mathcal{L})^{2m+1}u;u\big\rangle_{L^2(I)} \quad &\leqslant 2^{2m-2}\Big\{\sum_{q=0}^{m}\int_I(1-\nu^2)\Big|\frac{d}{d\nu}\Big(f_q(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu})^q u\Big)\Big|^2\,d\nu\\ &+\sum_{q=0}^{m-1}\int_I(1-\nu^2)\Big|\frac{d}{d\nu}\Big((1-\nu^2)g_q\frac{d}{d\nu}(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu})^q u\Big)\Big|^2\,d\nu\Big\}.\end{aligned}$$

Using the inequalities

$$\begin{aligned}\int_I(1-\nu^2)\Big|\frac{d}{d\nu}\Big(f_q(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu})^q u\Big)\Big|^2\,d\nu \leqslant \quad & 2\|(1-\nu^2)|\frac{df_q}{d\nu}|^2\|_{L^\infty(I)}\int_I\Big|(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu})^q u\Big|^2\,d\nu\\ & 2\|f_q^2\|_{L^\infty(I)}\int_I(1-\nu^2)\Big|(\frac{d}{d\nu}\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu})^q u\Big|^2\,d\nu\end{aligned}$$

and

$$\begin{aligned}&\int_I(1-\nu^2)\Big|\frac{d}{d\nu}\Big((1-\nu^2)g_q\frac{d}{d\nu}(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu})^q u\Big)\Big|^2\,d\nu \leqslant\\ &\qquad 2\|(1-\nu^2)|\frac{d}{d\nu}(1-\nu^2)g_q|^2\|_{L^\infty(I)}\int_I\Big|(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu})^q u\Big|^2\,d\nu\\ &\qquad 2\||(1-\nu^2)g_q|^2\|_{L^\infty(I)}\int_I(1-\nu^2)\Big|\frac{d}{d\nu}(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu})^q u\Big|^2\,d\nu\end{aligned}$$

we obtain the desired result.

∎

A.3.2 Continuity of B_2

Since we use the same arguments for proving the continuity of B_2, we won't give as much computational detail as in the preceding proof. However we begin with a preliminary remark. If $u \in C^0(I)$ then

$$\frac{d}{d\nu}\Big\{\int_{-1}^{+1}\int_0^1(ikt)^{2n}u(z)e^{ikt|\nu-z|}\,dz\,dt\Big\} = \int_{-1}^{+1}\int_0^1(ikt)^{2n+1}\mathrm{sign}(\nu-z)u(z)e^{ikt|\nu-z|}\,dz\,dt,$$

and

$$\begin{aligned}\frac{d}{d\nu}\Big\{\int_{-1}^{+1}\int_0^1(ikt)^{2n+1}\mathrm{sign}(\nu-z)u(z)e^{ikt|\nu-z|}\,dz\,dt\Big\} = \quad & \int_{-1}^{+1}\int_0^1(ikt)^{2n+2}u(z)e^{ikt|\nu-z|}\,dz\,dt\\ & +2u(\nu)\int_0^1(ikt)^{2n+1}dt.\end{aligned}$$

Lemma A.5
For $q=0\dots N$ suppose there exist $f_q, g_q, h_q^1, h_q^2 \in C^\infty(\overline{I})$ and $u \in E^{2N}(I)$ such that

$$\begin{aligned}v(\nu) = \quad & \sum_{q=0}^{N}f_q(\nu)\int_{-1}^{+1}\int_0^1(ikt)^{2q}u(z)e^{ikt|\nu-z|}\,dtdz\\ & +\sum_{q=0}^{N-1}g_q(\nu)\int_{-1}^{+1}\int_0^1(ikt)^{2q+1}u(z)\boldsymbol{sign}(\nu-z)e^{ikt|\nu-z|}\,dtdz\\ & +\sum_{q=0}^{N-1}h_q^1(\nu)\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^q u \;+\; \sum_{q=0}^{N-2}h_q^2(\nu)(1-\nu^2)\frac{d}{d\nu}\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^q u.\end{aligned}$$

Then there exist $F_q, G_q, H_q^1, H_q^2 \in C^\infty(\overline{I})$ $q = 0 \dots N$ *such that*

$$\begin{aligned}
\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}v(\nu) = \quad & \sum_{q=0}^{N+1} F_q(\nu)\int_{-1}^{+1}\int_0^1 (ikt)^{2q}u(z)e^{ikt|\nu-z|}\,dtdz \\
& + \sum_{q=0}^{N} G_q(\nu)\int_{-1}^{+1}\int_0^1 (ikt)^{2q+1}u(z)\boldsymbol{sign}(\nu-z)e^{ikt|\nu-z|}\,dtdz \\
& + \sum_{q=0}^{N} H_q^1(\nu)\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^q u \;+\; \sum_{q=0}^{N-1} H_q^2(\nu)(1-\nu^2)\frac{d}{d\nu}\Big(\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big)^q u.
\end{aligned}$$

Proof:

Again, we use recurrence arguments. For $N = 0$,

$$\begin{aligned}
&\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big\{f_0(\nu)\int_{-1}^{+1}\int_0^1 u(z)e^{ikt|\nu-z|}\,dtdz\Big\} = \\
&\qquad \Big(\frac{d}{d\nu}(1-\nu^2)\frac{df_0}{d\nu}\Big)\int_{-1}^{+1}\int_0^1 u(z)e^{ikt|\nu-z|}\,dtdz \\
&\qquad + 2\frac{d}{d\nu}\Big((1-\nu^2)f_0\Big)\int_{-1}^{+1}\int_0^1 (ikt)\mathrm{sign}(\nu-z)\,u(z)\,e^{ikt|\nu-z|}\,dtdz \\
&\qquad + (1-\nu^2)f_0(\nu)\int_0^1 (ikt)^2\,u(z)\,e^{ikt|\nu-z|}\,dtdz \\
&\qquad + (1-\nu^2)f_0(\nu)\,ik\,u(\nu).
\end{aligned}$$

so the result is true for $N = 0$. Suppose now that the result is true for $q = 0 \dots N$. According to the recurrence assumption, and using the preceding lemma, it is sufficient to prove that

$$\begin{aligned}
&(i)\ \frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big\{f_N(\nu)\int_{-1}^{+1}\int_0^1 (ikt)^{2N}u(z)e^{ikt|\nu-z|}\,dtdz\Big\} \\
&(ii)\ \frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big\{g_N(\nu)\int_{-1}^{+1}\int_0^1 (ikt)^{2N-1}u(z)\,\mathrm{sign}(\nu-z)\,e^{ikt|\nu-z|}\,dtdz\Big\}
\end{aligned}$$

have the expected form. We will show it only for (ii), since the proof for (i) would be very similar to the case $N = 0$.

$$\begin{aligned}
&\frac{d}{d\nu}(1-\nu^2)\frac{d}{d\nu}\Big\{\,g_N(\nu)\int_{-1}^{+1}\int_0^1 (ikt)^{2N-1}u(z)\,\mathrm{sign}(\nu-z)\,e^{ikt|\nu-z|}\,dtdz\Big\} = \\
&\qquad \Big(\frac{d}{d\nu}(1-\nu^2)\frac{dg_N}{d\nu}\Big)\int_{-1}^{+1}\int_0^1 (ikt)^{2N-1}u(z)\,\mathrm{sign}(\nu-z)\,e^{ikt|\nu-z|}\,dtdz \\
&\qquad + 2\,\frac{d}{d\nu}\Big((1-\nu^2)g_N\Big)\int_{-1}^{+1}\int_0^1 (ikt)^{2N}u(z)\,e^{ikt|\nu-z|}\,dtdz \\
&\qquad + 4\,\frac{d}{d\nu}\Big((1-\nu^2)g_N\Big)\int_0^1 (ikt)^{2N-1}dt\ u(\nu) \\
&\qquad (1-\nu^2)g_N(\nu)\int_{-1}^{+1}\int_0^1 (ikt)^{2N+1}u(z)\,\mathrm{sign}(\nu-z)\,e^{ikt|\nu-z|}\,dtdz \\
&\qquad + 2\,g_N(\nu)\int_0^1 (ikt)^{2N-1}dt\ (1-\nu^2)\frac{du}{d\nu}
\end{aligned}$$

We obtain the desired form.

■

Lemma A.6 $\forall n \in \mathbb{N}, \quad B_2 : E^n(I) \to E^n(I)$ *is continuous.*

Proof:
Note that
$$\int_{-1}^{+1} u(z)\,\frac{e^{ik|\nu-z|}-1}{|\nu-z|}\,dz = \int_{-1}^{+1}\int_{0}^{1} u(z)e^{ikt|\nu-z|}\,dzdt.$$
Applying the preceding lemma, we conclude like in the proof of the continuity of B_1.

■

A.3.3 Continuity of B_3

The continuity of B_3 will be far more easy to verify.

Lemma A.7 $\forall n \in \mathbb{N}$, $B_3 : E^n(I) \mapsto E^n(I)$ *is continuous.*

Proof:
Let us consider two points $\mathbf{x}, \mathbf{x}' \in J$ with respective ellipsoidal coordinates $(1;\nu;\theta)$ and $(1;\nu';\theta')$. Then we denote $\mathcal{L}_{\mathbf{x}} = \frac{\partial}{\partial\nu}(1-\nu^2)\frac{\partial}{\partial\nu}$. We have
$$\|B_3 u\|^2_{E^n(I)} = \langle(-\mathcal{L})^n B_3 u; B_3 u\rangle_{L^2(I)} \leqslant \|(-\mathcal{L})^n B_3 u\|_{L^2(I)}\,\|B_3 u\|_{L^2(I)}.$$
Moreover
$$(\mathcal{L})^q B_3 u = \int_J u(\mathbf{x}')\,(\mathcal{L}_{\mathbf{x}})^q G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}')\,d\mathbf{x}'.$$
So the Cauchy-Schwartz inequality yields
$$\|(\mathcal{L})^q B_3 u\|^2_{L^2(I)} \leqslant \int_{J\times J} |(\mathcal{L})^q G^{\mathrm{reg}}(\mathbf{x};\mathbf{x}')|^2\,d\mathbf{x}\,d\mathbf{x}'\;\|u\|^2_{L^2(I)}.$$
hence the desired result choosing $q = 0$ and $q = n$.

■

A.4 Density lemma

Lemma A.8
$H^1_{00}(\Omega_R \setminus J)$ *is dense in* $H^1(\Omega_R)$ *for the norm* $\|\;\|_{H^1(\Omega_R)}$.

Proof:
Since $C^\infty(\overline{\Omega}_R)$ is dense in $H^1(\Omega_R)$, it is sufficient to show that any $v \in C^\infty(\overline{\Omega}_R)$ is the limit in $H^1(\Omega_R)$ of a sequence belonging to $H^1_{00}(\Omega_R \setminus J)$. Consider such a v. We define a radiall function ($\mathbf{x}$ refers to the point with ellipsoidal coordinates $(\xi;\nu;\theta)$)
$$\chi_n(\mathbf{x}) = \begin{cases} 0 & \text{si } \xi^2 < 1+1/n \\[2mm] \dfrac{\ln(n(\xi^2-1))}{\ln n} & \text{si } 1+1/n < \xi^2 < 2 \\[2mm] 1 & \text{si } \xi^2 > 2. \end{cases}$$
We define $v_n = \chi_n\, v$. Then for any n, $v_n \in H^1_{00}(\Omega_R)$. Let us show that this sequence has the desired property.
$$\begin{aligned} \|v - v_n\|_{H^1(\Omega_R)} &\leqslant \|v - v_n\|_{L^2(\Omega_R)} + \|\nabla(v - v_n)\|_{L^2(\Omega_R)} \\ &\leqslant \|v\|_{L^\infty(\Omega_R)}\Big(\|1-\chi_n\|_{L^2(\Omega_R)} + \|\nabla\chi_n\|_{L^2(\Omega_R)}\Big) + \|\nabla v\|_{L^\infty(\Omega_R)}\|1-\chi_n\|_{L^2(\Omega_R)}. \end{aligned}$$

It is sufficient to prove that $\lim_{n\to+\infty} \|1-\chi_n\|_{L^2(\Omega_R)} = 0$ and $\lim_{n\to+\infty} \|\nabla\chi_n\|_{L^2(\Omega_R)} = 0$. Dominated convergence theorem shows that $\lim_{n\to+\infty} \|1-\chi_n\|_{L^2(\Omega_R)} = 0$. Moreover

$$\frac{\partial \chi_n}{\partial \xi} = \frac{1}{\ln n}\,\frac{2\xi}{\xi^2-1}.$$

hence the existence of a constant $\kappa > 0$ such that

$$\|\nabla\chi_n\|^2_{L^2(\Omega_R)} = \int_{\xi=\sqrt{1+1/n}}^{\sqrt{2}} \int_{\nu=-1}^{+1} \int_{\theta=0}^{2\pi} (\xi^2-1)|\frac{\partial \chi_n}{\partial \xi}|^2 d\xi d\nu d\theta \leqslant \frac{64\pi}{\ln^2 n} \int_{\sqrt{1+1/n}}^{\sqrt{2}} \frac{d\xi}{\xi^2-1} \leqslant \frac{\kappa}{\ln n}.$$

This concludes the proof since the right hand side of the inequality goes to 0 when $n \to \infty$. ■

A.5 Technical result of differential geometry

Lemma A.9
Consider $\mathcal{S}$ a C^∞ submanifold of $\mathbb{R}^3$ of dimension 2. Let U be an open subset of $\mathbb{R}^3$ such that (U,ϕ) is chart of $\mathbb{R}^3$ corresponding to the coordinates (x_1,x_2,x_3). Suppose that for a point $\mathbf{x} \in U$ with coordinates (x_1,x_2,x_3), $\mathbf{x} \in \mathcal{S} \Leftrightarrow x_1 = F(x_2,x_3)$, where F is a C^∞ function (which is another manner to say that $\mathcal{S}$ is C^∞). Finaly suppose that the metric tensor of $\mathbb{R}^3$ in the chart (U,ϕ) is given by $h_1^2 d^2x_1 + h_2^2 d^2 x_2 + h_3^2 d^2 x_3$. and denote $n = n_1\frac{\partial}{\partial x_1} + n_2\frac{\partial}{\partial x_2} + n_3\frac{\partial}{\partial x_3}$ the vector field normal to $\mathcal{S}$ such that $n_1 > 0$ and $\gamma dx_1 dx_2 dx_3$ the surface measure on $\mathcal{S}$. Then

$$\gamma\, n_1 = \frac{h_2 h_3}{h_1}.$$

Proof:

The proof consists in straightforward calculus. We explicitly compute n and γ. A basis for the tangent space of $\mathcal{S}$ is given by the two vector fields $\frac{\partial F}{\partial x_2}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ and $\frac{\partial F}{\partial x_3}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$. Thus we immediatly compute the expression of the metric tensor on $\mathcal{S}$ in the coordinates (x_2,x_3),

$$[\,g\,]^{\mathcal{S}}_{\phi} = \begin{bmatrix} h_1^2(\frac{\partial F}{\partial x_2})^2 + h_2^2 & h_1^2(\frac{\partial F}{\partial x_2}\frac{\partial F}{\partial x_3}) \\ h_1^2(\frac{\partial F}{\partial x_2}\frac{\partial F}{\partial x_3}) & h_1^2(\frac{\partial F}{\partial x_2})^2 + h_3^2 \end{bmatrix}.$$

We deduce

$$\begin{aligned}\gamma^2 &= \Big(h_1^2(\frac{\partial F}{\partial x_2})^2 + h_2^2\Big)\Big(h_1^2(\frac{\partial F}{\partial x_3})^2 + h_3^2\Big) - \Big(h_1^2(\frac{\partial F}{\partial x_2}\frac{\partial F}{\partial x_3})\Big)^2 = h_2^2h_3^2 + h_2^2h_1^2(\frac{\partial F}{\partial x_3})^2 + h_3^2h_1^2(\frac{\partial F}{\partial x_2})^2 \\ &= h_2^2h_3^2\Big(1 + (\frac{\partial F}{\partial x_2})^2\frac{h_1^2}{h_2^2} + (\frac{\partial F}{\partial x_3})^2\frac{h_1^2}{h_3^2}\Big).\end{aligned}$$

Besides, the normal field n is defined by the equations

$$n_1\frac{\partial F}{\partial x_2}h_1^2 + n_2h_2^2 = 0 \qquad n_1\frac{\partial F}{\partial x_3}h_1^2 + n_3h_3^2 = 0 \qquad n_1^2h_1^2 + n_2^2h_2^2 + n_3^2h_3^2 = 1$$

which yields

$$n_1^2h_1^2 + n_1^2(\frac{\partial F}{\partial x_2})^2\frac{h_1^4}{h_2^2} + n_1^2(\frac{\partial F}{\partial x_3})^2\frac{h_1^4}{h_3^2} = 1 \quad\Rightarrow\quad n_1^2 = \frac{1}{h_1^2}\,\frac{1}{1 + (\frac{\partial F}{\partial x_2})^2\frac{h_1^2}{h_2^2} + (\frac{\partial F}{\partial x_3})^2\frac{h_1^2}{h_3^2}}.$$

A combination of the preceding identities provides the conclusion of the lemma

$$n_1^2 = \frac{1}{h_1^2}\frac{h_2^2h_3^2}{\gamma^2} \quad\Rightarrow\quad \gamma^2 n_1^2 = \frac{h_2^2h_3^2}{h_1^2} \quad\Rightarrow\quad \gamma n_1 = \frac{h_2h_3}{h_1} \quad \text{since} \quad n_1 > 0.$$

■

Contents

References

[1] A.Mazari. *Détermination par une méthode d'équations intégrales du champ électromagnétique rayonné par une structure filiforme.* PhD thesis, Universit'e Paris VI, 1991.

[2] A.Taflove. *Computational electrodynamics.* Artech House Inc., 1995.

[3] D.S.Jones. *Methods in Electromagnetic Wave Propagation.* Oxford Engineering Science Series. Oxford Science Publications, 1994.

[4] J.Sanchez-Hubert et E.Sanchez-Palencia. *Introduction aux méthodes asymptotiques et a l'homogénéisation.* Masson, 1992.

[5] F.Collino et F.Millot. Fils et méthodes d' éléments finis pour les équations de maxwell. le modèle de holland revisit'e. Technical Report 3472, Inria, 1998.

[6] F.Brezzi et M.Fortin. *Mixed and Finite Element Methods.* Springer-Verlag, 1991.

[7] G.Vial et S.Tordeux. Matching of asymptotic expansions and multiscale expansion for the rounded corner problem. Technical report, SAM, ETH, Zürich, 2006.

[8] M.V. Fedoryuk. Asymptotics of the solution of the dirichlet problem for the laplace and helmholtz equations in the exterior of a slender cylinder. *Izv. Akad. Nauk SSSR Ser. Mat.*, 1981.

[9] F.Rogier. *Problèmes mathématiques et numériques lies à l'approximation de la géométrie d'un corps diffractant dans les équations de l'électromagnétisme.* PhD thesis, Paris VI, 1989.

[10] G.F.Maslennikova. A neumann problem for the helmholtz operator in the exterior to a thin body of revolution. *Differential equations*, 20 (2):316–324, 1984.

[11] G.Szegö. *Orthogonal Polynomials.* American Mathematical Society, 1939.

[12] G.V.Zhdanova. Dirichlet problem for the helmholtz operator in the exterior of a thin body of revolution. *Differential Equations*, 20 (8):1403–1411, 1984.

[13] H.C.Pocklington. Electrical oscillations in wires. *Pro. of the Cambridge Philosophical Society*, 1897.

[14] Il'in. *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems*, volume 102 of *Translation of Mathematical Monographs.* American Mathematical Society, 1992.

[15] J-C.Nedelec. *Acoustic and Electromagnetic Equations.* Springer, 2001.

[16] Marc Lenoir. Equations intégrales et problèmes de diffraction, Cours ENSTA, 2004-2005.

[17] L.Robin. *Fonctions sphériques de Legendre et fonctions sphéroïdales.* Gauthier-Villars, 1957.

[18] M.V.Fedoryuk. The dirichlet problem for the laplace operator in the exterior of a thin body of revolution. In *Theory of Cubature Formulas and the Applications of Functionnal Analysis to Problems of Mathematical Physics*, number 2 in 126. American Mathematical Society Translations, 1985.

[19] N.N.Lebedev. *Special Functions and their applications.* Dover Publications, 1972.

[20] F.W.J Olver. *Asymptotics and special functions.* Computer Science and Applied Methematics. Academic Press, 1974.

[21] R.Holland and L.Simpson. Finite-difference analysis of emp coupling to thin struts and wires. *IEEE Trans. Electromagn. Compat.*, 1981.

[22] S.Tordeux. *Méthodes asymptotiques pour la propagation des ondes dans les milieux comportant des fentes.* PhD thesis, UVSQ-Universite de Versailles Saint-Quentin-en-Yvelines, 2004.

[23] G. Vial. *Analyse multi-échelle et conditions aux limites avec couche mince dans un domaine à coin.* PhD thesis, Université de Rennes 1, 2003.

[24] W.Eckhaus. *Asymptotic analysis of singular perturbations.* North-Holland, 1979.

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399